On the general One-Dimensional XY Model: positive and zero temperature, selection and non-selection

A. T. Baraviera(*), L. M. Cioletti(**), A. O. Lopes (*),
J. Mohr (*) and R. R. Souza(*)

November 2, 2011

Abstract

We consider (M, d) a connected and compact manifold and we denote by $\mathcal{B}_i$ the Bernoulli space $M^{\mathbb{Z}}$. The analogous problem on the half-line $\mathbb{N}$ is also considered. Let $A : \mathcal{B}_i \to \mathbb{R}$ be an *observable*. Given a temperature T, we analyze the main properties of the Gibbs state $\hat{\mu}_{\frac{1}{T}A}$.

In order to do our analysis we consider the Ruelle operator associated to $\frac{1}{T}A$, and we get in this procedure the main eigenfunction $\psi_{\frac{1}{T}A}$. Later, we analyze selection problems when the temperature goes to zero: a) existence, or not, of the limit $V := \lim_{T \to 0} T \log(\psi_{\frac{1}{T}A})$, a question about selection of subactions, and, b) existence, or not, of the limit $\tilde{\mu} := \lim_{T \to 0} \hat{\mu}_{\frac{1}{T}A}$, a question about selection of measures.

The existence of subactions and other properties of Ergodic Optimization are also considered.

The case where the potential depends just on the coordinates (x_0, x_1) is carefully analyzed. We show, in this case, and under suitable hypotheses, a Large Deviation Principle, when $T \to 0$, graph properties, etc... Finally, we will present in detail a result due to A. C. D. van Enter and W. M. Ruszel, where the authors show, for a particular example of potential A, that the selection of measure $\hat{\mu}_{\frac{1}{T}A}$ in this case, does not happen.

0 Introduction

Let (M, d) be a connected and compact manifold. We denote by $\mathcal{B}$ the Bernoulli space $M^{\mathbb{N}}$ of sequences represented by $x = (x_0, x_1, x_2, x_3, ...)$, where $x_i, i \geq 0$ belongs to the space (alphabet) M. By Tychonoff´s Theorem of compactness, we know $\mathcal{B}$ is a compact metric space when equipped with the distance given by $d_c(x, y) = \sum_{k \geq 0} \frac{d(x_k, y_k)}{c^k}$, with $c > 1$. The topologies generated by d_{c_1} or d_{c_2} are the same. We denote d when we choose $c = 2$. In several of our results M is the interval $[0, 1]$ or the one-dimensional circle $\mathbb{S}^1$.

The shift σ on $\mathcal{B}$ is defined by $\sigma((x_0, x_1, x_2, x_3, ...)) = (x_1, x_2, x_3, x_4, ...)$. It is a continuous function on $\mathcal{B}$.

Let $A : \mathcal{B} \to \mathbb{R}$ be an *observable* or *potential* defined on the Bernoulli space $\mathcal{B}$, i.e. a real-valued function defined on $\mathcal{B}$. The potential A describes an interaction between sites in the one-dimensional lattice $M^{\mathbb{N}}$.

For most of the results we consider here we will require A to be Hölder-continuous, which means there exist constants $0 < \alpha < 1$ and $Hol_A > 0$ such that $|A(x) - A(y)| \leq Hol_A d(x,y)^\alpha$. We call α the exponent of A and Hol_A the constant for A. We will be interested here in the Gibbs state μ_A associated to such A, which will be a probability measure on $\mathcal{B}$. Note that the set of probability measures on $\mathcal{B}$ is compact for the weak* topology, (which is given by a metric).

For each value $\beta = 1/T$, where T is the temperature, we can consider the Gibbs state $\mu_{\beta A}$, and, we want to show in a particular example (introduced by A. C. D. van Enter and W. M. Ruszel [20]), that there is no limit (in the weak* topology) of the family $\mu_{\beta A}$, when $\beta \to \infty$. We will present here in section 6 all the details of the proof of this non-trivial result.

We point out that by trivial modification of the metric a Holder potential can be considered a Lipschitz potential (with no change of the topology). Therefore, we can state our results in either case. The assumption of A being Lipschitz means that there is fast rate of decay of influence of the potential if we are far away in the lattice.

The case of the lattice $\mathbb{Z}$, that is $\mathcal{B}_i = M^{\mathbb{Z}}$ can be treated in a similar way: Let $A : M^{\mathbb{Z}} \to \mathbb{R}$ be a Lipschitz potential, and denote by $\hat{\sigma}$ the left-shift on $\mathcal{B}_i$. Any Lipschitz potential on $\mathcal{B}_i$ is $\hat{\sigma}$-cohomologous to a potential on $\mathcal{B}$ (same proof as in Proposition 1.2 [45] or, in [7]). We will explain this more carefully later. To consider $\hat{\sigma}$-invariant probability measures on $\mathcal{B}_i$ means that the position $0 \in \mathbb{Z}$ in the lattice is not distinguished (which in general makes sense).

We call general one-dimensional XY model the setting described above. A particularly interesting case is when we consider $M = \mathbb{S}^1$ (the unit one-dimensional circle) [22] [35] [20]. This one-dimensional continuous Ising model is another important example that can be treated in the setting. Below in section 1 our results are for the general case of any M as above.

We say that the potential $A : \mathcal{B} \to \mathbb{R}$ depends on the first two coordinates if $A(x) = A(x_0, x_1, x_2, ..) = A(x_0, x_1)$, for any $x = (x_0, x_1, x_2, ..)$. In this case A is always Lipschitz. Such kind of potentials are sometimes called nearest neighbor interaction potentials. The so-called one-dimensional XY model in most of the cases assumes that A depends on the first two coordinates [22]. Special attention to this case will be given in section 4. For example, in [22] [19]

$$A(x) = A(x_0, x_1) = \cos(x_1 - x_0 - \alpha) + \gamma \cos(2\,x_0),$$

where α and γ are constants. The part $\gamma \cos(2\,x_0)$ corresponds to the magnetic term while $\cos(x_1 - x_0 - \alpha)$ corresponds to the interaction term.

We point out that this point of view of getting a coboundary and the systematic use of the Ruelle operator is the Thermodynamical Formalism setting (see [45]). This, in principle, is different from the point of view more commonly used in Statistical Mechanics on general lattices where the Gibbs measures are defined by means of a specification, DLR formalism, limit of probabilities on finite boxes (see [27], [19], [44]). We briefly address this question for a potential which depends on two coordinates in section 5.

In the Classical Thermodynamic Formalism one usually considers $M = \{1, 2, ..., d\}$ [45] [33]. Here M is a compact manifold with a volume form. We point out that we will use the following notation: we call a **Gibbs** probability measure for A the measure which is derived from a Ruelle operator, and we call the **equilibrium** probability measure for A the one which is derived from a maximization of Pressure (which requires one to be able to talk about entropy). We will be interested here in **Gibbs** states because we need to avoid to talk about entropy. Note that the shift acting on $M^{\mathbb{N}}$ is such that each point has an uncountable number of pre-images. Just in some late sections we will speak about "entropy" and "pressure" of the potential A (in general in the case it depends on two coordinates).

An interesting discussion about the several possible approaches (DLR, Thermodynamic limit in finite boxes, etc..) to Statistical Mechanics in the one-dimensional lattice appears in [53].

Some of the results presented here will be used in a future related paper [40].

We point out that the understanding of Statistical Mechanics via the Ruelle Operator (Transfer Operator) allows one to get eigen-functions, and, in the limit (in the logarithm scale), when temperature goes to zero, the subaction. This helps in getting Large Deviation properties of Gibbs states when temperature goes to zero [3] [40].

In the first part of this paper we describe the theory for case of general A (section 1 for positive temperature and section 2 for zero temperature). Later (in section 4) we will focus on the case where the potential A depends only on the first two coordinates. Section 5 compares the setting of Thermodynamical Formalism with DLR Formalism. These two sections will help a better understanding of Section 6 where we present a detailed explanation of an example [20] where there is no selection of measures.

1 Positive temperature: a generalized Ruelle-Perron-Frobenius Theorem

Let $\mathcal{C}$ be the space of continuous functions from $\mathcal{B} = M^{\mathbb{N}}$ to $\mathbb{R}$. We are interested in the Ruelle operator on $\mathcal{C}$ associated to the Lipschitz observable $A : M^{\mathbb{N}} \to \mathbb{R}$, which acts on $\psi \in \mathcal{C}$, and sends it to $\mathcal{L}_A(\psi) \in \mathcal{C}$ defined by

$$\mathcal{L}_A(\psi)(x) = \int_M e^{A(ax)} \psi(ax) \, da \, ,$$

for any $x = (x_0, x_1, x_2, ...) \in \mathcal{B}$, where ax represents the sequence $(a, x_0, x_1, x_2, ...) \in \mathcal{B}$, and da is the Lebesgue probability measure on M. Note that $\sigma(ax) = x$.

A major difference between the settings of the Classical Bowen-Ruelle-Sinai Thermodynamic Formalism setting and the XY model is that here, in order to define the Ruelle operator, we need an a priori measure (for which we consider in most of the cases the Lebesgue probability measure da on $\mathbb{S}^1$).

Some of the results of the present section are generalization of theorems in [38].

The operator $\mathcal{L}_A$ will help us to find the Gibbs state for A. First we will show the existence of a main eigenfunction for $\mathcal{L}_A$, when A is Lipschitz. Part of our proof

follows the reasoning of section 7 in [1] (which considers $M = \{1, 2, .., d\}$), adapted to the present case.

We begin by defining another operator on $\mathcal{C}$. Let $0 < s < 1$, and define, for $u \in \mathcal{C}$, $\mathcal{T}_{s,A}(u)$ given by

$$\mathcal{T}_{s,A}(u)(x) = \log\left(\int_M e^{A(ax)+su(ax)}\, da\right).$$

Proposition 1. *If* $0 < s < 1$ *then* $\mathcal{T}_{s,A}$ *is an uniform contraction map.*

Proof.:

$$|\mathcal{T}_{s,A}(u_1)(x) - \mathcal{T}_{s,A}(u_2)(x)| = \left|\log\left(\frac{\int_M e^{A(ax)+su_1(ax)}}{\int_M e^{A(ax)+su_2(ax)}}\right)\right| =$$

$$= \left|\log\left(\frac{\int_M e^{A(ax)+su_2(ax)+su_1(ax)-su_2(ax)}}{\int_M e^{A(ax)+su_2(ax)}}\right)\right| \le$$

$$\le \log\left(\frac{\int_M e^{A(ax)+su_2(ax)+s\|u_1-u_2\|}}{\int_M e^{A(ax)+su_2(ax)}}\right) = s\|u_1 - u_2\|.$$

$\square$

Let u_s be the unique fixed point for $\mathcal{T}_{s,A}$. We have

$$\log\left(\int_M e^{A(ax)+su_s(ax)}\, da\right) = u_s(x). \tag{1}$$

Proposition 2. *The family* $\{u_s\}_{0<s<1}$ *is an equicontinuous family of functions.*

Proof.: Let $H_s(x, y) = u_s(x) - u_s(y)$. By (1) we have

$$
\begin{aligned}
e^{u_s(x)} &= \int_M e^{A(ax)+su_s(ax)} \\
&= \int_M e^{A(ay)+su_s(ay)} e^{A(ax)-A(ay)+s[u_s(ax)-u_s(ay)]} \\
&\le e^{u_s(y)} \max_a \{e^{A(ax)-A(ay)+s[u_s(ax)-u_s(ay)]}\}.
\end{aligned}
$$

Hence

$$e^{u_s(x)-u_s(y)} \le \max_a\{e^{A(ax)-A(ay)+s[u_s(ax)-u_s(ay)]}\},$$

and this implies

$$H_s(x, y) = u_s(x) - u_s(y) \le \max_a[A(ax) - A(ay) + sH_s(ax, ay)].$$

Proceeding by induction we get

$$H_s(x, y) \le \max_{\theta \in \mathcal{B}} \sum_{n=0}^{\infty} s^n[A(\theta_n....\theta_0 x) - A(\theta_n...\theta_0 y)] \le$$

4

$$\leq Hol_A \max_{\theta \in \mathcal{B}} \sum_{n=0}^{\infty} s^n d((\theta_n....\theta_0 x), (\theta_n...\theta_0 y))^{\alpha} \leq$$

$$\leq Hol_A \sum_{n=0}^{\infty} \left(\frac{s}{2^{\alpha}}\right)^n d(x,y)^{\alpha} \leq \frac{2^{\alpha}}{2^{\alpha}-1} Hol_A d(x,y)^{\alpha}.$$

Remark 1: This shows that u_s is Lipschitz, and, moreover, that u_s, $0 \leq s < 1$, is an equicontinuous family. Note the very important point: the Lipschitz constant of u_s, is given by $\frac{2^{\alpha}}{2^{\alpha}-1} Hol_A$, and depends only on the Holder constant for A, but does not depend on s.

$\square$

Let

$$S_n(z) = S_{n,A}(z) = \sum_{k=0}^{n-1} A \circ \sigma^k(z).$$

Note that iterates of the operator $\mathcal{L}_A$ can be written with the use of $S_{n,A}(z)$.

$$\mathcal{L}_A^n(w)(x) = \int_{\mathbf{a} \in M^n} e^{S_{n,A}(\mathbf{a}x)} w(\mathbf{a}x) \, d\mathbf{a}.$$

Theorem 3. *There exists a strictly positive Lipschitz eigenfunction ψ_A for $\mathcal{L}_A : \mathcal{C} \to \mathcal{C}$ associated to a strictly positive eigenvalue λ_A. The eigenvalue is simple and it is equal to the spectral radius.*

Proof. It follows from the fixed point equation that for any x

$$-||A|| + s \min u_s \leq u_s(x) \leq ||A|| + s \max u_s.$$

Therefore, $-||A|| \leq (1-s) \min u_s \leq (1-s) \max u_s \leq ||A||$, for any s. Consider a subsequence $s_n \to 1$ such that $[(1-s_n) \max u_{s_n}] \to k$.

The family $\{u_s^* = u_s - \max u_s\}_{0 < s < 1}$ is equicontinuous and uniformly bounded.

Therefore, by Arzela-Ascoli $\{u_{s_n}^*\}_{n \geq 1}$ has an accumulation point in $\mathcal{C}$, which we will call u.

Observe that for any s

$$e^{u_s^*(x)} = e^{u_s(x) - \max u_s} =$$

$$e^{-(1-s) \max u_s + u_s(x) - s \max u_s} =$$

$$e^{-(1-s) \max u_s} \int e^{A(ax) + (s u_s(ax) - s \max u_s)} \, da.$$

Taking limit where n goes to infinity for the sequence s_n we get that u satisfies

$$e^{u(x)} = e^{-k} \int e^{A(ax) + u(ax)} \, da.$$

In this way we get a **positive** Lipschitz eigenfunction $\psi_A = e^u$ for $\mathcal{L}_A$ associated to the eigenvalue $\lambda_A = e^k$.

$\square$

Remark 2: To prove that u is Lipschitz, we just use the fact that u is the limit of a sequence of uniformly Lipschitz functions (i.e. Lipschitz functions with same

Lipschitz constant). Using that u is a bounded function we have that $\psi_A = e^u$ is also Lipschitz. Note a very important point: the Lipschitz constant of $u = \log(\psi_A)$ is given by $\frac{2^\alpha}{2^\alpha - 1} Hol_A$(see Remark 1 in the end of the proof of Proposition 2).

The property that the eigenvalue is simple and maximal follows from the same reasoning as in page 23 and 24 of [45]. For example, to prove that the eigenvalue is simple we suppose there are two eigenfunctions ψ_1 and ψ_2. Let $t = \min\{\psi_1/\psi_2\}$. Then $\psi_3 = \psi_1 - t\psi_2$ is a non-negative eigenfunction which vanishes at some point $z \in \mathcal{B}$. Therefore

$$0 = \lambda_A^n \psi_3(z) = \int_{\mathbf{a} \in M^n} e^{S_{n,A}(\mathbf{a}z)} \psi_3(\mathbf{a}z) \, d\mathbf{a} \,,$$

which implies $\psi_3(\mathbf{a}z) = 0 \ \forall \, \mathbf{a} \in M^n$, $\forall n$, which makes $\psi_3 = 0$.

$\square$

Note that

$$\int_M \frac{e^{A(ax)}\psi_A(ax)}{\lambda_A \psi_A(x)} da = 1 \,, \ \forall x \in \mathcal{B} \,. \tag{2}$$

If a potential B satisfies

$$\int_M e^{B(ax)} da = 1 \,, \ \forall x \in \mathcal{B} \,,$$

which means $\mathcal{L}_B(1) = 1$, we say that B is normalized.

Let

$$\bar{A} = A + \log \psi_A - \log \psi_A \circ \sigma - \log \lambda_A,$$

where $\sigma : \mathcal{B} \to \mathcal{B}$ is the usual shift map. Equation (2) shows that $\bar{A}$ is normalized. It is also Lipschitz (Holder). In this case the main eigenvalue is 1 and the main eigenfunction is constant equal to 1 (in fact we can prove, using proposition 4, that there is only one strictly positive eigenfunction, the one associated to the maximal eigenvalue).

Remember that, given $x = (x_0, x_1, x_2, ...) \in \mathcal{B}$ and $a \in M$, we denote by $ax \in \mathcal{B}$ the element $ax = (a, x_0, x_1, x_2, ...)$, i.e., any $y \in \mathcal{B}$ such that $\sigma(y) = x$ is of this form.

We define the Borel sigma-algebra $\mathcal{F}$ over $\mathcal{B}$ as the σ-algebra generated by the *cylinders*. By this we mean the sigma-algebra generated by sets of the form $B_1 \times B_2 \times ... \times B_n \times M^{\mathbb{N}}$, where $n \in \mathbb{N}$, and $B_j, j \in \{1, 2, .., n\}$, are open sets in M. Similar definitions can be considered for $\mathcal{B}_i$.

We say a probability measure μ over $\mathcal{F}$ is invariant, if for any Borel set B, we have that $\mu(B) = \mu(\sigma^{-1}(B))$. This corresponds to stationary probability measures for the underlying stochastic process X_n, $n \in \mathbb{N}$, with state space M. We denote by $\mathcal{M}_\sigma$ the set of invariant probability measures. Similar definitions can be considered for $\mathcal{B}_i$.

We present below a generalization of results considered in [45].

We define the dual operator $\mathcal{L}_A^*$ on the space of the Borel measures on $\mathcal{B}$ as the operator that sends a measure v to the measure $\mathcal{L}_A^*(v)$ defined by

$$\int_{\mathcal{B}} \psi \, d\mathcal{L}_A^*(v) = \int_{\mathcal{B}} \mathcal{L}_A(\psi) \, dv \,.$$

for any $\psi \in \mathcal{C}$.

Now we want to find an eigen-probability for $\mathcal{L}_A^*$. This will help us to find the Gibbs state for the potential A.

Proposition 4. *If the observable $\bar{A}$ is normalized, then there exists an unique fixed point $m = m_{\bar{A}}$ for $\mathcal{L}_{\bar{A}}^*$. Such a probability measure m is σ-invariant, and for all Holder continuous function ω we have that, in the uniform convergence topology,*

$$\mathcal{L}_{\bar{A}}^n \omega \to \int_{\mathcal{B}} \omega dm \,.$$

Here $\mathcal{L}_{\bar{A}}^n$ denotes the n-th iterate of the operator $\mathcal{L}_{\bar{A}} : \mathcal{C} \to \mathcal{C}$.

Proof.: We begin by proving that the normalization property implies that the convex and compact set of Borel probability measures on $\mathcal{B}$ is preserved by the operator $\mathcal{L}_{\bar{A}}^*$: in order to see that, note that for μ a Borel probability measure on $\mathcal{B}$, we have

$$\mathcal{L}_{\bar{A}}^*(\mu)(\mathcal{B}) = \int_{\mathcal{B}} 1 \ d\mathcal{L}_{\bar{A}}^*(\mu) = \int_{\mathcal{B}} \mathcal{L}_{\bar{A}}(1) d\mu = \int_{\mathcal{B}} 1 \ d\mu = \mu(\mathcal{B}) = 1$$

where the third equality is precisely the normalization hypothesis.

By the Tychonoff-Schauder theorem let m be a fixed point for the operator $\mathcal{L}_{\bar{A}}^*$.

To prove that m is σ-invariant, we begin by observing that

$$\mathcal{L}_{\bar{A}}(\psi \circ \sigma)(x) = \int_M e^{\bar{A}(ax)} \psi \circ \sigma(ax) da = \int_M e^{\bar{A}(ax)} \psi(x) da = \psi(x).$$

Note that the normalization hypothesis is used in the last equality.

Therefore, if $\psi \in \mathcal{C}$, then

$$\int_{\mathcal{B}} \psi \circ \sigma dm = \int_{\mathcal{B}} \psi \circ \sigma d\mathcal{L}_{\bar{A}}^*(m) = \int_{\mathcal{B}} \mathcal{L}_{\bar{A}}(\psi \circ \sigma) dm = \int_{\mathcal{B}} \psi dm.$$

which implies the invariance property of m.

Before finishing the proof of proposition 4, we will need two claims. The first is a special estimate which will be important in the rest of this section.

Claim: For any Holder potential A, if $\|w\|$ denotes the uniform norm of the Holder function $w : \mathcal{B} \to \mathbb{R}$, we have

$$|\mathcal{L}_A^n(w)(x) - \mathcal{L}_A^n(w)(y)| \leq \left[C_{e^A} \|w\| \left(\frac{1}{2^\alpha} + ... + \frac{1}{2^{n\alpha}} \right) + \frac{C_w}{2^{n\alpha}} \right] d(x, y)^\alpha,$$

where C_{e^A} is the Holder constant of e^A and C_w is the Holder constant of w.

Proof of the Claim: : We prove the claim by induction. Suppose $n = 1$. We have

$$|\mathcal{L}_A(w)(x) - \mathcal{L}_A(w)(y)| \leq$$

$$\leq \int_M |e^{A(ax)} - e^{A(ay)}| \cdot |w(ax)| da + \int_M e^{A(ay)} |w(ax) - w(ay)| da \leq$$

$$\leq (C_{e^A} \|w\| + C_w) \frac{d(x,y)^\alpha}{2^\alpha},$$

where in the last inequality we used the normalization property of A. In particular we can say that the Holder constant of $\mathcal{L}_A(w)$ is given by

$$C_{\mathcal{L}_A(w)} = \frac{C_{e^A} \|w\| + C_w}{2^\alpha}. \tag{3}$$

Now, suppose the Claim holds for n. We have

$$|\mathcal{L}_A^{n+1}(w)(x) - \mathcal{L}_A^{n+1}(w)(y)| = |\mathcal{L}_A^n(\mathcal{L}_A(w))(x) - \mathcal{L}_A^n(\mathcal{L}_A(w))(y)| \leq$$

$$\leq \left[C_{e^A} \|\mathcal{L}_A(w)\| \left(\frac{1}{2^\alpha} + ... + \frac{1}{2^{n\alpha}} \right) + \frac{C_{\mathcal{L}_A(w)}}{2^{n\alpha}} \right] d(x,y)^\alpha,$$

and, therefore the claim is proved when we use (3) and $\|\mathcal{L}_A(w)\| \leq \|w\|$ which is consequence of the normalization property of A.

As a consequence, the set $\{\mathcal{L}_{\bar{A}}^n \omega\}_{n \geq 0}$ is equicontinuous. In order to prove that $\{\mathcal{L}_{\bar{A}}^n \omega\}_{n \geq 0}$ is uniformly bounded we use again the normalization condition which implies $\|\mathcal{L}_{\bar{A}}^n \omega\| \leq \|w\|, \forall n \geq 1$.

By the Arzela-Ascoli Theorem let $\bar{\omega}$ be an accumulation point for $\{\mathcal{L}_{\bar{A}}^n \omega\}_{n \geq 0}$, i.e., suppose there exists a subsequence $\{n_k\}_{k \geq 0}$ such that

$$\bar{\omega}(x) = \lim_{k \geq 0} \mathcal{L}_{\bar{A}}^{n_k} \omega(x).$$

Second Claim: : $\bar{\omega}$ is a constant function.
The proof of this second claim is similar to the reasoning of page 25 [45].
Now that $\bar{\omega}$ is a constant function we can prove that

$$\bar{\omega} = \int_{\mathcal{B}} \bar{\omega} dm = \lim_k \int_{\mathcal{B}} \mathcal{L}_{\bar{A}}^{n_k} \omega dm = \lim_k \int_{\mathcal{B}} \omega d(\mathcal{L}_{\bar{A}}^*)^{n_k}(m) = \int_{\mathcal{B}} \omega dm,$$

which shows that $\bar{\omega}$ does not depend on the subsequence chosen. Therefore, for any $x \in \mathcal{B}$ we have

$$\mathcal{L}_{\bar{A}}^n \omega(x) \to \bar{\omega} = \int_{\mathcal{B}} \omega dm.$$

The last limit shows that the fixed point m is unique.

$\square$

Proposition 5. *Let A be a Holder, not necessarily normalized potential, and ψ_A and λ_A the eigenfunction and eigenvalue given by theorem 3. To the potential A we associate the normalized potential $\bar{A} = A + \log \psi_A - \log \psi_A \circ \sigma - \log \lambda_A$. Let m be the unique probability measure that satisfies $\mathcal{L}_{\bar{A}}^*(m) = m$, given by proposition 4.*
(a) the measure

$$\rho_A = \frac{1}{\psi_A} m$$

8

satisfies $\mathcal{L}_A^*(\rho_A) = \lambda_A \rho_A$. Therefore, ρ_A is an eigen-probability for $\mathcal{L}_A^*$.

(b) for any Holder $\phi : \mathcal{B} \to \mathbb{R}$, we have that

$$\frac{\mathcal{L}_A^n(\phi)}{\lambda_A^n} \to \psi_A \int \phi \, d\rho_A.$$

Proof: (a) $\mathcal{L}_{\bar{A}}^*(m) = m$ implies that for any $\psi \in \mathcal{C}$, we have

$$\int \psi dm = \int \psi d\mathcal{L}_{\bar{A}}^*(m)$$

$$= \int \mathcal{L}_{\bar{A}}(\psi) dm$$

$$= \int \left(\int \psi(ax) e^{\bar{A}(ax)} da \right) dm(x)$$

$$= \int \left(\int \psi(ax) \frac{e^{A(ax)} \psi_A(ax)}{\lambda_A \psi_A(x)} da \right) dm(x).$$

Now, if $\varphi \in \mathcal{C}$, making $\psi = \frac{\varphi}{\psi_A}$ in the last equation we have

$$\int \frac{\varphi}{\psi_A} dm = \frac{1}{\lambda_A} \int \left(\int \varphi(ax) \frac{e^{A(ax)}}{\psi_A(x)} da \right) dm(x),$$

which is equivalent to

$$\lambda_A \int \varphi d\rho_A = \int \mathcal{L}_A(\varphi) d\rho_A \tag{4}$$

or

$$\mathcal{L}_A^*(\rho_A) = \lambda_A \rho_A.$$

(b) We have that $A = \bar{A} - \log \psi_A + \log \psi_A \circ \sigma + \log \lambda_A$, and therefore

$$S_{n,A}(z) \equiv \sum_{k=0}^{n-1} A \circ \sigma^k(z) = S_{n,\bar{A}}(z) - \log \psi_A + \log \psi_A \circ \sigma^n + n \log \lambda_A,$$

which makes

$$\frac{\mathcal{L}_A^n(\phi)(x)}{\lambda_A^n} = \frac{1}{\lambda_A^n} \int_{\mathbf{a} \in M^n} e^{S_{n,A}(\mathbf{a}x)} \phi(\mathbf{a}x) d\mathbf{a} =$$

$$= \psi_A(x) \int_{\mathbf{a} \in M^n} \frac{e^{S_{n,\bar{A}}(\mathbf{a}x)}}{\psi_A(\mathbf{a}x)} \phi(\mathbf{a}x) d\mathbf{a} =$$

$$= \psi_A(x) \mathcal{L}_{\bar{A}}^n \left(\frac{\phi}{\psi_A} \right) \to \psi_A(x) \int \frac{\phi}{\psi_A} dm_{\bar{A}}$$

where the convergence on n in the last line comes from Proposition 4.   □

Remark 3: From now on we will call $m_{\bar{A}}$ the eigen-probability for $\mathcal{L}_{\bar{A}}^*$. One can show that the eigen-probability $\rho_A = \frac{1}{\psi_A} m_{\bar{A}}$ is the unique eigen-probability for $\mathcal{L}_A^*$. Also, it is not necessarily invariant for the shift σ.

9

We call $m_{\bar{A}}$ the Gibbs state for A. This probability measure $m_{\bar{A}}$ over $\mathcal{B}$ is invariant for the shift and describes the statistics of the interaction described by A. It is usual to call the probability measure $m_{\bar{A}}$ the Gibbs state (in the Thermodynamic Formalism setting [45]) for the interaction given by A.

We point out that the probability measure ρ_A is positive on open sets of $\mathcal{B}$. Suppose the metric space $M = \mathbb{S}^1$. The projection of this probability measure on the first two coordinates $\mathbb{S}^1 \times \mathbb{S}^1$ is absolutely continuous with respect to Lebesgue probability measure on $\mathbb{S}^1 \times \mathbb{S}^1$. This is so because, if B is Borel in $[0,1]^2$, then from (4) we have

$$\int I_{(x_0,x_1)\in B}\, d\rho_A = \frac{1}{\lambda_A^2} \int \mathcal{L}_{\bar{A}}^2 \left(I_{(x_0,x_1)\in B} \right) d\rho_A,$$

and, for any $x \in \mathcal{B}$

$$\mathcal{L}_{\bar{A}}^2 (I_{(x_0,x_1)\in B})(x) = \int_M \int_M e^{S_{2,\bar{A}}\,(abx)} I_{(x_0,x_1)\in B}(abx)\, da\, db.$$

Remark 4: If we consider instead a Holder potential $B : \mathcal{B}_i = M^{\mathbb{Z}} \to \mathbb{R}$, where

$$\mathcal{B}_i = \{(..., x_{-2}, x_{-1}, x_0, x_1, x_2, ...)|\, x_i \in M, i \in \mathbb{Z}\},$$

then, we first derive (as in Proposition 1.2 [45] or, in [7]) the associated cohomologous Holder potential $A : \mathcal{B} \to \mathbb{R}$ (the Holder class can change), then proceed as above to get ρ_A over $\mathcal{B}$. Finally, we consider the natural extension $\hat{\rho}_A$ of ρ_A on $\mathcal{B}_i$ (see [46] [7]), and we solve in this way the Statistical Mechanics problem for the interaction described by B in the lattice $\mathbb{Z}$: it´s the probability measure $\hat{\rho}_{\beta A}$.

Note that if C is a set that depends just on the coordinates x_0, x_1, then $\rho_{\beta A}(C) = \hat{\rho}_{\beta A}(C)$. For sets $C \subset \mathcal{B}_i$, of this form, we can use without loss of generality $\rho_{\beta A}(C)$ or $\hat{\rho}_{\beta A}(C)$.

Proposition 6. *The only Lipschitz continuous eigenfunction ψ of $\mathcal{L}_A$ which is totally positive is ψ_A (the one associated to the maximal eigenvalue λ_A).*

Proof: Suppose $\psi : \mathcal{B} \to \mathbb{R}$ is a Lipschitz continuous eigenfunction of $\mathcal{L}_A$ associated to the eigenvalue β.

It follows from the above that $\frac{\mathcal{L}_A^n(\psi)}{\lambda_A^n} \to \psi_A \int \psi d\rho_A$, when $n \to \infty$.

Therefore, if $\psi > c > 0$, then $\int \psi d\rho_A > 0$. Moreover, $\mathcal{L}_A^n(\psi) = \beta^n \psi$. This is only possible if $\beta = \lambda_A$ and $\psi = \psi_A$.

$\square$

It is easy to see that if A is Holder with exponent α, and, denoting $\mathcal{H}_\alpha$, the set of real-valued functions with Holder exponent α, then $\mathcal{L}_{\bar{A}} : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$.

For $w \in \mathcal{H}_\alpha$, denote $|w|_\alpha = \sup_{x \neq y} \frac{|w(x)-w(y)|}{d(x,y)^\alpha}$. It is known that $\mathcal{H}_\alpha$ is a Banach space for the norm

$$||w||_\alpha = |w|_\alpha + ||w||,$$

where $||w||$ is the uniform norm of w.

When $\alpha = 1$ we are considering the space of Lipschitz functions $\mathcal{H}_1$.

We note that $\mathcal{K}_\alpha \equiv \{w \in \mathcal{H}_\alpha, \, ||w||_\alpha \leq 1\}$ is compact in the uniform norm as a subset of $\mathcal{C}$. To prove that, we just need to observe that the definition of the norm $||w||_\alpha$ implies that $\mathcal{K}_\alpha$ is a equicontinuous and uniformly bounded set, and then we have the result directly by using Arzela-Ascoli´s theorem.

We can also prove that $\mathcal{K}_\alpha^A \equiv \{w \in \mathcal{H}_\alpha, \, \int_w dm_A = 0, \, ||w||_\alpha \leq 1\}$ is compact in the uniform norm. For doing that, let $I_{m_A} : \mathcal{H}_\alpha \to \mathbb{R}$ be given by $I_{m_A}(w) = \int w\,dm_A$. We have that I_{m_A} is a bounded linear operator, and therefore $I_{m_A}^{-1}\{0\}$ is a closed subset of $\mathcal{H}_\alpha$. Now $\mathcal{K}_\alpha^A = \mathcal{K}_\alpha \cap I_{m_A}^{-1}\{0\}$ is compact.

Proposition 7. *Suppose $\bar{A}$ is normalized, then the eigenvalue $\lambda_{\bar{A}} = 1$ is maximal. Moreover, the remainder of the spectrum of $\mathcal{L}_{\bar{A}} : \mathcal{H}_\alpha \to \mathcal{H}_\alpha$ is contained in a disk centered at zero with radius strictly smaller than one.*

Proof. Remember that 1 is the eigenfunction associated to the eigenvalue 1. We will show that $\mathcal{L}_{\bar{A}}$ restricted to $\mathcal{K}_\alpha^{\bar{A}}$ has spectral radius strictly smaller than 1. We know from proposition 4 that $\mathcal{L}_{\bar{A}}^k$ converges to zero in the compact set $\mathcal{K}_\alpha^{\bar{A}}$.

The normalization hypothesis implies $||\mathcal{L}_{\bar{A}}^{n+1}(w)|| \leq ||\mathcal{L}_{\bar{A}}^n(w)|| \,\forall n \geq 0$. We will now prove that this monotonicity property implies that the convergence above is uniform. More precisely, we have

Claim: Given a small ϵ, there exists $N = N_\epsilon \in \mathbb{N}$ such that

$$||\mathcal{L}_{\bar{A}}^n(w)|| < \epsilon \,\forall\, n \geq N, \, \forall w \in \mathcal{K}_\alpha^{\bar{A}}.$$

To prove this claim, let $C_n \equiv \{w \in \mathcal{K}_\alpha^{\bar{A}} : ||\mathcal{L}_{\bar{A}}^m(w)|| < \epsilon \,\forall m \geq n\}$. The monotonicity property implies $C_n \subseteq C_{n+1}$ and also that C_n is an open set in the uniform norm, while $\mathcal{L}_{\bar{A}}^k(w) \to 0$ implies $\cup_n C_n = \mathcal{K}_\alpha^{\bar{A}}$. Therefore, compactness of $\mathcal{K}_\alpha^{\bar{A}}$ implies $\mathcal{K}_\alpha^{\bar{A}} = C_N$ for some $N \in \mathbb{N}$.

The last claim is easy to prove and can be enunciated as:

Claim: There exists $C > 0$ such that

$\forall n \in \mathbb{N}$ and $w \in \mathcal{H}_\alpha$

$$|\mathcal{L}_{\bar{A}}^n(w)|_\alpha \leq C||w|| + \frac{|w|_\alpha}{(2^\alpha)^n}.$$

Now, for any given n and k, using the last Claim we have for $w \in \mathcal{H}_\alpha$

$$|\mathcal{L}_{\bar{A}}^{n+k}(w)|_\alpha \leq C||\mathcal{L}_{\bar{A}}^k(w)|| + \frac{|\mathcal{L}_{\bar{A}}^k(w)|_\alpha}{(2^\alpha)^n} \leq C||\mathcal{L}_{\bar{A}}^k(w)|| + C\frac{||w||}{(2^\alpha)^n} + \frac{|w|_\alpha}{(2^\alpha)^{n+k}}.$$

Therefore, if ϵ is small enough and $n \geq N_\epsilon$, we have that for all $w \in \mathcal{K}_\alpha^{\bar{A}}$

$$||\mathcal{L}_{\bar{A}}^{n+k}(w)||_\alpha \leq \epsilon < 1.$$

In this case the spectral radius is smaller than $\epsilon^{\frac{1}{n+k}}$.

$\square$

We denote $\lambda_{\bar{A}}^1 < \lambda_{\bar{A}}$ the spectral radius of $\mathcal{L}_{\bar{A}}$ when restricted to the set $\{w \in \mathcal{H}_\alpha : \int w\,dm_{\bar{A}} = 0\}$.

Now we will show the exponential decay of correlation for Holder functions.

Proposition 8. *If $v, w \in \mathcal{L}^2(m_{\bar{A}})$ are such that w is Holder and $\int w \, dm_{\bar{A}} = 0$, then, there exists $C > 0$ such that for all n*

$$\int (v \circ \sigma^n) \, w \, dm_{\bar{A}} \leq C \, (\lambda_{\bar{A}}^1)^n$$

Proof. This follows from

$$\int (v \circ \sigma^n) \, w \, dm_{\bar{A}} = \int v \, \mathcal{L}_{\bar{A}}^n w \, dm_{\bar{A}}.$$

$\square$

The above proposition implies that $m_{\bar{A}}$ is mixing (same reasoning as in section 2 of [33] which considers the case of the shift on $\{1, 2, .., d\}^{\mathbb{N}}$).

Proposition 9. *The invariant probability measure $m_{\bar{A}}$ is ergodic.*

Proof. If a dynamical system is mixing then it is ergodic (see section 2 in [33]).

$\square$

A major difference of the general XY Model to the Thermodynamic Formalism setting (in the sense of [45] [33]) in $\{1, 2, ..., d\}^{\mathbb{Z}}$ is that here we can not define in the traditional way (via dynamic partitions) the concept of entropy of an invariant probability measure μ (defined over the sigma algebra $\mathcal{F}$ of $\mathcal{B}$). Each element $x \in \mathcal{B}$ has an uncountable set of pre-images and this is a problem.

Note that there exist invariant probabilities (for instance, singular with respect to Lebesgue measure) for the shift on $\mathcal{B}$ which have Kolmogorov entropy arbitrarily large.

For the other hand, in the DLR-Gibbs theory, see [31], a definition of entropy is presented and the variational principle at positive temperatures is worked out and proved. But here we take another path, just in terms of transfer operators, and we present the theory of the Ruelle operator for continuous-spin models, and also including some noncontinuous potentials, which is not part of standard treatments.

Note that the Gibbs state formalism via boundary conditions, as in [27], does not require, in principle, to talk about entropy (see also our Section 5). We will address the question about entropy when the potential depends on two coordinates in section 4.

In Statistical Mechanics, for a fixed interaction A under a certain temperature $T > 0$, up to a multiplicative constant, the natural potential to be considered is $\frac{1}{T} A$. We denote $\beta = \frac{1}{T}$, and, using the results above we can consider the corresponding eigenfunction $\psi_{\beta A}$, eigenvalue $\lambda_{\beta A} = \lambda_\beta$, and the Gibbs state which now will be denoted $\mu_{\beta A}$.

What happen with these two objects when $T \to 0$ (or, $\beta \to \infty$), is the purpose of the next section.

2   Zero temperature: calibrated subactions, maximizing probability measures and selection of probability measures

In this section and also in the next two sections we will consider, among other issues, questions involving selections of probability measures when the temperature goes to zero, maximizing probability measures for a given potential and existence of calibrated subactions. Among other results we will show that, under some conditions, the sequence $\{\mu_{\beta A}\}$ of Gibbs states for the potential βA converges to a measure μ_∞ which has the property of maximizing the integral $\int A d\mu$ among all invariant measures μ for the shift map. Sometimes such convergence will not occur (this is what we call non selection of probability measures - a very interesting example due to A. C. D. van Enter and W. M. Ruszel will be presented in section 6).

We will also consider calibrated subactions, which is an important tool that allows one to identify the support of the maximizing probability measure μ_∞ (see equation (6) below), and can be used to relate the maximal eigenvalues of the Ruelle operator to the value $m(A) = \int A d\mu_\infty$ (see theorem 11). Existence of calibrated subactions are also related to the existence of large deviation principles for the convergence of $\{\mu_{\beta A}\}$ to μ_∞ (see theorem 18 in section 4).

Some of the problems discussed here are usually called ergodic optimization problems (see [32]). We refer the reader to [16] for question related to Ergodic Transport Theory.

Consider a fixed Holder potential A and a real variable $\beta > 0$. We denote by $\psi_{\beta A}$ the eigenfunction for the Ruelle operator associated to βA.

Remark 5: Given β and A, the Lipschitz constant of u_β, such that $\psi_{\beta A} = e^{u_\beta}$, depends on the Holder constant for βA (see Remarks 1 and 2). More precisely, the Lipschitz constant of $u_\beta = \log(\psi_{\beta A})$ is given by $\beta \frac{2^\alpha}{2^\alpha - 1} Hol_A$. Therefore, $\frac{1}{\beta} \log(\psi_{\beta A})$, $\beta > 0$, is equicontinuous. Note that it is also uniformly bounded from the reasons described below.

A possible renormalization condition for $\psi_{\beta A}$ [15] is $\int \psi_{\beta A} d\rho_{\beta A} = 1$, where $\rho_{\beta A}$ is the eigen-probability for $\mathcal{L}_{\beta A}^*$ (see proposition 5 and remark 3). For each $\beta > 0$ the normalization hypothesis $\int \psi_{\beta A} d\rho_{\beta A} = 1$ implies the existence of $x_\beta \in \mathcal{B}$ such that $\psi_\beta(x_\beta) = 1$. Here we are using the connectedness hypothesis of $\mathcal{B}$. When $\beta \to \infty$ we have that $x_{\beta_k} \to \bar{x}$, for a subsequence. Note that when we normalize $\psi_{\beta A}$ the Holder constant of $\log(\psi_{\beta A})$ remains unchanged, which assures the uniformly continuous property of the family $1/\beta \log(\psi_{\beta A})$, $\beta > 0$. Moreover, the normalization hypothesis and Remark 5 implies that $1/\beta \log(\psi_{\beta A})$, $\beta > 0$ is uniformly bounded.

Therefore, there exists a subsequence $\beta_n \to \infty$, and V Lipschitz, such that on the uniform convergence

$$V := \lim_{n \to \infty} \frac{1}{\beta_n} \log(\psi_{\beta_n A}).$$

Consider point $p_0 \in \mathcal{B}$. Another possible normalization for the eigenfunction $\psi_{\beta A}$ is to assume that $\psi_{\beta A}(p_0) = 1$. We will prefer this latter form.

By selection of a function V, when the temperature goes to zero (or, $\beta \to \infty$), we mean the existence of the limit (in the uniform norm)

$$V := \lim_{\beta \to \infty} \frac{1}{\beta} \log(\psi_{\beta A}).$$

The existence of the limit when $\beta \to \infty$ (not just of a subsequence), in the general case, is not an easy question.

In this section we denote $\mu_{\beta A}$ the Gibbs state for the potential βA, i.e. the eigen-probability of $\mathcal{L}_{\bar{A}}^*$, where $\bar{A} = A + \log \psi_A - \log \psi_A \circ \sigma - \log \lambda_A$.

By selection of a measure $\tilde{\mu}_\infty$, when the temperature goes to zero (or, $\beta \to \infty$), we mean the existence of the limit (in the weak* sense)

$$\tilde{\mu}_\infty := \lim_{\beta \to \infty} \mu_{\beta A}.$$

In some sense V is what one can get in the limit, in the log-scale, from the eigen-function (at non-zero temperature), and $\tilde{\mu}_\infty$ is the Gibbs state at temperature zero.

Even if A is Lipschitz not always the above limit on $\mu_{\beta A}$, $\beta \to \infty$, exist. In fact we will show an interesting example in section 6 (due to A. C. D. van Enter and W. M. Ruszel) where there is no limit for $\mu_{\beta A}$, as $\beta \to \infty$.

Some theorems in this section are generalizations of corresponding ones in [38] (which consider only potentials A which depend on two coordinates). Related results appear in [25] and [26]. Results about selection (or, non selection) in the setting of Thermodynamic Formalism appear in [5] [4] [9] [36] [8] [40].

Some of the proofs and results presented in the present section are similar to other ones in Ergodic Optimization [32] and Thermodynamic Formalism, but the main point is that we have to avoid in the proofs the concept of entropy and the variational principle of pressure.

Remember that we denote by $\mathcal{M}_\sigma$ the set of σ invariant Borel probability measures over $\mathcal{B}$. As $\mathcal{M}_\sigma$ is compact, given A, there always exists a subsequence β_n, such that $\mu_{\beta_n A}$ converges to an invariant probability measure.

We consider the following problem: given $A : \mathcal{B} \to \mathbb{R}$ Lipschitz, we want to find measures that maximize, over $\mathcal{M}_\sigma$, the value

$$\int A(x)\, d\mu(\mathbf{x}).$$

We define

$$m(A) = \max_{\mu \in \mathcal{M}_\sigma} \left\{ \int A d\mu \right\}.$$

Any of these measures will be called a maximizing probability measure, which is sometimes denoted by μ_∞. As $\mathcal{M}_\sigma$ is compact, there exist always at least one maximizing probability measure. It is also true that there exists ergodic maximizing probability measures. Indeed, the set of maximizing probability measures is convex,

compact and the extreme probability measures of this convex set are ergodic (can not be expressed as convex combination of others [33]). Any maximizing probability measure is a convex combination of ergodic ones [46].

Even when A is Holder the maximizing probability measure μ_∞ do not have to be unique. For instance, suppose that A is Holder and has maximum value just in the union of two different fixed points (for the shift σ) $p_0 \in \mathcal{B}$ and $p_1 \in \mathcal{B}$. In this case the set of maximizing probability measures μ_∞ is $\{t\,\delta_{p_0} + (1-t)\delta_{p_1}\,|\,t \in [0,1]\}$.

Note that δ_{p_0} and δ_{p_1} are ergodic, but the other maximizing probability measures are not.

Similar definitions for a potential $A : \mathcal{B}_i \to \mathbb{R}$ and maximization of $\int A\,d\hat{\mu}$, over all the $\hat{\mu}$ which are $\hat{\sigma}$-invariant probability measures, can also be considered. Questions about selection of measure also make sense.

Definition 1. *A continuous function* $u : \mathcal{B} \to \mathbb{R}$ *is called a* calibrated subaction *for* $A : \mathcal{B} \to \mathbb{R}$, *if, for any* $y \in \mathcal{B}$, *we have*

$$u(y) = \max_{\sigma(x)=y} [A(x) + u(x) - m(A)]. \tag{5}$$

This can also be expressed as

$$m(A) = \max_{a \in M} \{A(ay) + u(ay) - u(y)\}.$$

Note that for any $x \in \mathcal{B}$ we have

$$u(\sigma(x)) - u(x) - A(x) + m(A) \geq 0.$$

The above equation for u can be seen as a kind of discrete version of a sub-solution of the Hamilton-Jacobi equation [12] [6] [21]. It can be also seen as a kind of dynamic additive eigenvalue problem [13] [14] [24].

If u is a calibrated subaction, then $u + c$, where c is a constant, is also a calibrated subaction. An interesting question is when such calibrated subaction u is unique up to an additive constant.

Remember that if ν is invariant for σ, then for any continuous function $u : \mathcal{B} \to \mathbb{R}$ we have

$$\int [u(\sigma(x)) - u(x)]\,d\nu = 0$$

Suppose μ is maximizing for A and u is a calibrated subaction for A.

It follows at once (see for instance [15] [32] [51] for a similar result) that for any x in the support of μ_∞ we have

$$u(\sigma(x)) - u(x) - A(x) + m(A) = 0. \tag{6}$$

In this way if we know the value $m(A)$, then a calibrated subaction u for A helps us to identify the support of maximizing probabilities. The above equation can be true outside the union of the supports of the maximizing probabilities.

Maximizing probability measures are natural candidates for being selected by $\mu_{\beta A}$, as $\beta \to \infty$. But, in our setting, without the maximizing principle of pressure (which one can take advantage of the classical Thermodynamic Formalism) this is not so obvious. We address the question in section 3.

15

Proposition 10. *For any β, we have $-\|A\| < \frac{1}{\beta} \log \lambda_\beta < \|A\|$.*

Proof: Fix $\beta > 0$. We choose $\bar{x}$ the maximum of $\psi_{\beta A}$ in $\mathcal{B}$ and $\tilde{x}$ the minimum of $\psi_{\beta A}$ in $\mathcal{B}$. Now, if $\|A\|$ is the uniform norm of A, we have

$$\lambda_\beta = \frac{1}{\psi_{\beta A}(\bar{x})} \int e^{\beta A(a\,\bar{x})} \psi_{\beta A}(a\,\bar{x}) da \leq \int e^{\beta A(a\,\bar{x})} da \leq e^{\beta \|A\|} \text{ and}$$

$$\lambda_\beta = \frac{1}{\psi_{\beta A}(\tilde{x})} \int e^{\beta A(a\,\tilde{x})} \psi_{\beta A}(a\,\tilde{x}) da \geq \int e^{\beta A(a\,\tilde{x})} da \geq e^{-\beta \|A\|} \, ,$$

which proves the result. $\qquad\qquad\square$

From now on, we will suppose $M = \mathbb{S}^1$ to avoid technical issues. But we claim that the following results hold for more general connected and compact manifolds.

Considering a subsequence β_n we get the existence of a limit $\frac{1}{\beta_n} \log \lambda_{\beta_n} \to K$, when $n \to \infty$. By taking a subsequence we can assume that is also true that there exists V Lipschitz, such that $V := \lim_{n\to\infty} \frac{1}{\beta_n} \log(\psi_{\beta_n A})$.

Given $y \in \mathcal{B}$, consider the equation

$$\lambda_{\beta_n} = \frac{1}{\psi_{\beta_n A}(y)} \int e^{\beta_n A(a\,y)} \psi_{\beta_n A}(a\,y) da.$$

It follows from Laplace method that, when $\beta \to \infty$,

$$K = \max_{a\in\mathbb{S}^1}\{A(ay) + V(ay) - V(y)\}.$$

If we are able to show that $K = m(A)$, then we can say that any limit of subsequence $\lim_{n\to\infty} \frac{1}{\beta_n} \log(\psi_{\beta_n A})$ is a calibrated subaction, and we will get, finally, that

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log \lambda_{\beta\,A} = \lim_{\beta\to\infty} \frac{1}{\beta} \log \lambda_\beta = m(A).$$

Next theorem is inspired by Theorem 1 in [1] and Theorem 3.3 in [28]. It follows from the last part of its proof that $K = m(A)$.

Theorem 11. *Given A Lipschitz there exists u Lipschitz which is a calibrated subaction for A. As a consequence, we have that*

$$\lim_{\beta\to\infty} \frac{1}{\beta} \log \lambda_\beta = m(A).$$

Proof. Suppose $A : \mathcal{B} \to \mathbb{R}$ is Lipschitz.

Given $0 < \lambda \leq 1$, consider the operator $\hat{\mathcal{L}}_\lambda : \mathcal{C} \to \mathcal{C}$ given by,

$$\hat{\mathcal{L}}_\lambda(u)(x) = \sup_{a\in\mathbb{S}^1}[A(ax) + \lambda u(ax)].$$

Given $x \in \mathcal{B}$, we denote by $a_x \in \mathbb{S}^1$ one of the points a where the supremum is attained.

It is easy to see that for any $0 < \lambda < 1$, the transformation $\hat{\mathcal{L}}_\lambda$ is a contraction on $\mathcal{C}$ with the uniform norm. Indeed, given $x \in \mathcal{B}$

$$\sup_{a \in \mathbb{S}^1} [A(ax) + \lambda u(ax)] - \sup_{b \in \mathbb{S}^1} [A(bx) + \lambda v(bx)] \leq$$

$$[A(a_x\, x) + \lambda u(a_x\, x)] - [A(a_x\, x) + \lambda v(a_x\, x)] \leq$$

$$\lambda u(a_x\, x) - \lambda v(a_x\, x) \leq \lambda ||u - v||.$$

Denote by u_λ the corresponding fixed point in $\mathcal{C}$. We want to show that u_λ is equicontinuous. Consider $x_0, y_0 \in \mathcal{B}$. For the given x_0 we take the corresponding $a_{x_0} \in M$, and then the we get $x_1 = a_{x_0} x_0$. By induction, given x_j, get $x_{j+1} = a_{x_j} x_j$.

We can also can get a sequence $y_j \in \mathcal{B}$, $j \geq 1$, such that, $y_j = a_{x_{j-1} \ldots} a_{x_1} a_{x_0} y_0$. Note that for all j we have $\sigma^j(y_j) = y_0$.

As for any j we have $u_\lambda(y_j) \geq A(y_{j+1}) - \lambda u_\lambda(y_{j+1})$, then

$$u_\lambda(x_j) - u_\lambda(y_j) \leq$$

$$[A(x_{j+1}) - A(y_{j+1})] + \lambda [u_\lambda(x_{j+1}) - u_\lambda(y_{j+1})].$$

Therefore, given x_0, y_0

$$u_\lambda(x_0) - u_\lambda(y_0) \leq \sum_{j=0}^{\infty} \lambda^j [A(x_j) - A(y_j)] \leq$$

$$(1 - \lambda) \sum_{j=0}^{\infty} \lambda^j \sum_{i=0}^{j} [A(x_i) - A(y_i)] \leq$$

$$\sup_j \sum_{i=0}^{j} [A(x_i) - A(y_i)] \leq$$

$$||A|| \sup_j \sum_{i=0}^{j} \left(\frac{1}{2}\right)^j d(x_0, y_0) < ||A|| \, 2 \, d(x_0, y_0).$$

This shows that u_λ is Lipschitz, and, moreover, that u_λ, $0 \leq \lambda < 1$, is an equicontinuous family. Note the very important point: the Lipschitz constant of u_λ depends on $||A||$.

Denote $u_\lambda^* = u_\lambda - \max u_\lambda$. Using Arzela-Ascoli we get the existence of a subsequence $\lambda_n \to 1$ such that $u_{\lambda_n}^* \to u$.

We claim that u is a subaction.

Indeed, given $x \in \mathcal{B}$, as $|u_\lambda(x)| \leq \lambda |u_\lambda(a_x\, x)| + |A(a_x\, x)| \leq \lambda ||u_\lambda|| + ||A(x)||$, then $(1 - \lambda)||u_\lambda|| < C$, where C is a constant.

From this follows that there is a constant k, such for some subsequence (of the previous subsequence λ_n), which will be also denoted by λ_n, we have $(1 - \lambda_n)||u_{\lambda_n}|| \to k$.

Note that for any λ

$$u_\lambda^*(x) = u_\lambda(x) - \max u_\lambda =$$

17

$$-(1-\lambda)\max u_\lambda + u_\lambda(x) - \lambda \max u_\lambda =$$

$$-(1-\lambda)\max u_\lambda + \max_{a\in\mathbb{S}^1}\{A(ax) + (\lambda u_\lambda(ax) - \lambda \max u_\lambda)\}.$$

Taking the limit n to infinity for the sequence λ_n we get

$$u(x) = -k + \max_{a\in\mathbb{S}^1}\{A(ax) + u(ax)\} = \max_{a\in\mathbb{S}^1}\{A(ax) + u(ax) - k\}.$$

Now, all we have to show is that $k = m(A)$.
From the above it follows at once that

$$-u(\sigma(y)) + u(y) + A(y) \le k.$$

If ν is a σ-invariant probability measure, then,

$$\int A(y)d\nu(y) = \int [u(\sigma(y)) - u(y) + A(y)]\, d\nu(y) \le k,$$

and, this shows that $m(A) \le k$.

Now we show that $m(A) \ge k$. Note that for any x there exist $y = a_x x$ such that $\sigma(y) = x$, and
$$-u(\sigma(y)) + u(y) + A(y) = k.$$

Therefore, the compact set $K = \{y\,|\,-u(\sigma(y)) + u(y) + A(y) = k\}$ is such that, $K' = \cap_n \sigma^{-n}(K)$ is non-empty, compact and σ-invariant. If we consider an σ-invariant probability measure ν with support on K', we have that $\int A(y)d\nu(y) = k$. From this follows that $m(A) \ge k$.

$\square$

Now we state a general result assuming just that A is continuous (not necessarily Lipschitz). We refer the reader to Theorem 1 in [23], Proposition 4 in [38], Theorem 2.4 in [28] for related results.

Theorem 12. *Given a potential $A \in \mathcal{C}$, we have*

$$m(A) = \inf_{f\in\mathcal{C}} \max_{(\mathbf{a},\mathbf{x})\in\mathbb{S}^1\times\mathcal{B}} [A(\mathbf{a}\,\mathbf{x}) + f(a\mathbf{x}) - f(\mathbf{x}))].$$

Proof: First, consider the convex correspondence $F : \mathcal{C} \to \mathbb{R}$ defined by $F(g) = \max(A + g)$. Consider also the subset

$$\mathcal{G} = \{g \in \mathcal{C} : \text{there exists } f \text{ such that } g(\mathbf{ax}) = f(a\,\mathbf{x}) - f(\mathbf{x}), \quad f \in \mathcal{C}\} \ne \emptyset.$$

Now consider the concave correspondence $G : \mathcal{C} \to \mathbb{R} \cup \{-\infty\}$ taking $G(g) = 0$, if $g \in \bar{\mathcal{G}}$, and $G(g) = -\infty$ otherwise.

Let $\mathcal{S}$ be the set of the signed measures over the Borel sigma-algebra of $\mathcal{B}$. Remember that the corresponding Fenchel transforms, $F^* : \mathcal{S} \to \mathbb{R} \cup \{+\infty\}$ and $G^* : \mathcal{S} \to \mathbb{R} \cup \{-\infty\}$, are given by

$$F^*(\hat{\mu}) = \sup_{g\in\mathcal{C}} \left[\int g(\mathbf{ax})\, d\hat{\mu}(\mathbf{ax}) - F(g) \right], \quad \text{and}$$

18

$$G^*(\hat{\mu}) = \inf_{g \in \mathcal{C}} \left[\int g(\mathbf{ax}) \, d\hat{\mu}(\mathbf{ax}) - G(g) \right].$$

Denote

$$\mathcal{S}_0 = \left\{ \hat{\mu} \in \mathcal{S} : \int f(\mathbf{ax}) \, d\hat{\mu}(\mathbf{ax}) = \int f(\mathbf{x}) \, d\hat{\mu}(\mathbf{x}) \, , \, \forall \, f \in \mathcal{C} \right\}.$$

We denote by $\mathcal{M}$ the set of probability measures over $\mathcal{B}$.

Given F and G as above, we claim that

$$F^*(\hat{\mu}) = \begin{cases} -\int_{\hat{\Sigma}} A(\mathbf{y}, \mathbf{x}) \, d\hat{\mu}(\mathbf{y}, \mathbf{x}) & \text{if } \hat{\mu} \in \mathcal{M} \\ +\infty & \text{otherwise} \end{cases} \quad \text{and}$$

$$G^*(\hat{\mu}) = \begin{cases} 0 & \text{if } \hat{\mu} \in \mathcal{S}_0 \\ -\infty & \text{otherwise} \end{cases}.$$

We refer the reader to the [23] or [38] for a proof of this claim (which is basically the same as we need here).

Once the correspondence F is Lipschitz, the theorem of duality of Fenchel-Rockafellar [47] assures

$$\sup_{g \in \mathcal{C}} \left[G(g) - F(g) \right] = \inf_{\hat{\mu} \in \mathcal{S}} \left[F^*(\hat{\mu}) - G^*(\hat{\mu}) \right].$$

$$\sup_{g \in \mathcal{G}} \left[- \max_{(\mathbf{a}, \mathbf{x}) \in \mathbb{S}^1 \times \mathcal{B}} (A + g)(\mathbf{ax}) \right] = \inf_{\hat{\mu} \in \mathcal{M}_\sigma} \left[- \int A(\mathbf{ax}) \, d\hat{\mu}(\mathbf{ax}) \right].$$

Finally, from the definition of $\mathcal{G}$, the claim of the theorem follows.

$\square$

3   A definition of entropy for Gibbs states at positive temperature and selection of probability measure

Given a Lipschitz function A we have that

$$\int \frac{e^{A(ax)} \psi_A(ax)}{\lambda_A \psi_A(x)} da = 1 \, , \, \forall x \in \mathcal{B} \, .$$

We denote as before

$$\bar{A} = A + \log \psi_A - \log \psi_A \circ \sigma - \log \lambda_A,$$

where $\sigma : \mathcal{B} \to \mathcal{B}$ is the usual shift map. In this case the normalized potential $\bar{A}$ satisfies

$$\int_M e^{\bar{A}(ax)} da = 1 \, , \, \forall x \in \mathcal{B} \, ,$$

which means $\mathcal{L}_{\bar{A}}(1) = 1$.

Therefore,

$$\int_{\mathcal{B}} \left[\int_M e^{\bar{A}(ax)} \, da \right] d\,\mu_A(x) = 1.$$

Note that for a fixed x the value $\bar{A}(ax)$ can not be smaller than zero for all $a \in M$. This is quite different from the analogous case where we consider the shift over $\{1, 2.., d\}^{\mathbb{N}}$ in the classical Thermodynamic Formalism.

For each $a \in M$, $x \in \mathcal{B}$, we denote by $J(ax) = \min\{1, e^{\bar{A}(ax)}\}$.

Definition 2. *Given the invariant probability measure μ_A, associated to the Lipschitz potential A, we define the entropy of μ_A as*

$$h(\mu_A) = -\int \log J(y) \, d\mu_A(y) > 0.$$

In other words

$$h(\mu_A) = -\int \bar{A}(y) \, I_{\{\bar{A} \le 0\}} (y) \, d\mu_A(y).$$

The set of probability measures μ_A, with A Lipschitz, is dense in the set of σ-invariant probability measures [37].

Note that μ_A is σ-invariant

$$-h(\mu_A) = \int \log J(y) \, d\mu_A(y) \le$$

$$\int \log \left(\frac{e^{A(y)} \psi_A(y)}{\lambda_A \psi_A(\sigma(y))} \right) d\mu_A(y) = \int A d\mu_A - \log \lambda_A.$$

Therefore,

$$\log \lambda_A \le h(\mu_A) + \int A d\mu_A.$$

For a fixed A consider now for each real value β the corresponding potential βA. Therefore,

$$\log \lambda_{\beta A} \le h(\mu_{\beta A}) + \beta \int A \, d\mu_{\beta A}.$$

Suppose for a certain subsequence β_n we have that $\mu_{\beta_n A} \to \mu$.

If we divide the last inequality by β_n, and, taking limit in n, we get

$$m(A) \le \limsup_{n \to \infty} \frac{h(\mu_{\beta_n A})}{\beta_n} + \int A \, d\mu.$$

From the above we can derive:

Theorem 13. *Suppose that $\mu = \lim_{n \to \infty} \mu_{\beta_n A}$, for some subsequence β_n, and*

$$\limsup_{n \to \infty} \frac{h(\mu_{\beta_n A})}{\beta_n} = 0,$$

then, the limit measure μ is a maximizing probability measure.

Corollary 14. *If the maximizing probability measure μ_∞ for A is unique, and,*

$$\limsup_{\beta \to \infty} \frac{h(\mu_{\beta A})}{\beta} = 0,$$

then, $\mu_{\beta A}$, when $\beta \to \infty$, selects the maximizing probability measure μ_∞.

4 Analysis of the case in which the potential depends on two coordinates

In this section we suppose the potential depends on two coordinates and the metric space is $M = \mathbb{S}^1$. In this case the Ruelle operator has a simple form. We will make the usual identification of $\mathbb{S}^1$ with $[0,1]$ (in further sections we will make the identification of $\mathbb{S}^1$ with $[0, 2\pi]$). We will present several results from [38] which will be needed in future sections.

We will need to define the following operators:

Definition 3. *Let $\mathcal{L}_\beta, \bar{\mathcal{L}}_\beta : C([0,1]) \to C([0,1])$ be given by*

$$\mathcal{L}_\beta \psi(y) = \int e^{\beta A(x,y)} \, \psi(x) dx, \tag{7}$$

$$\bar{\mathcal{L}}_\beta \psi(x) = \int e^{\beta A(x,y)} \, \psi(y) dy. \tag{8}$$

We refer the reader to [34] and [49] chapter IV for general results on positive integral operators. The next theorem (Krein-Ruthman) is well known. It will follow that, when A depends just on two coordinates (x_0, x_1), then the eigenfunction of the Ruelle operator (as defined in previous sections) depends only on the first coordinate x_0 (similar to [55]).

Theorem 15. *The operators $\mathcal{L}_\beta$ and $\bar{\mathcal{L}}_\beta$ have the same positive maximal eigenvalue λ_β, which is simple and isolated. The eigenfunctions associated are positive functions.*

Let us call $\psi_\beta, \bar{\psi}_\beta$ the positive eigenfunctions for $\mathcal{L}_\beta$ and $\bar{\mathcal{L}}_\beta$ associated to λ_β, which satisfy the normalization condition $\int \psi_\beta(x)\, dx = 1$ and $\int \bar{\psi}_\beta(x)\, dx = 1$.

We will define a density $\theta_\beta : [0,1] \to \mathbb{R}$ by

$$\theta_\beta(x) := \frac{\psi_\beta(x)\, \bar{\psi}_\beta(x)}{\pi_\beta}, \tag{9}$$

where $\pi_\beta = \int \psi_\beta(x)\bar{\psi}_\beta(x)dx$, and a transition $K_\beta : [0,1]^2 \to \mathbb{R}$ by

$$K_\beta(x,y) := \frac{e^{\beta A(x,y)}\, \bar{\psi}_\beta(y)}{\bar{\psi}_\beta(x)\, \lambda_\beta}\,. \tag{10}$$

The above expressions are consistent with the results obtained in [22] section 3. This can be formulated also as a variational pressure problem as we will see soon.

Note that if $A(x, y) = A(y, x)$, then ψ_β and $\bar{\psi}_\beta$ are constant, and, therefore θ_β is constant equal to 1. This happen for $M = \mathbb{S}$ in the case A is of the form $U(x - y)$ for a periodic function U. This case will be consider later.

Consider a probability measure ν on $[0, 1]^2$ that can be disintegrated as $d\nu(x, y) = d\theta(x)dK_x(y)$, where $\theta : [0, 1] \to [0, +\infty)$ and $K : [0, 1]^2 \to [0, +\infty)$ are continuous functions. We will denote this by $\nu = \theta K$, where θ is a continuous density of probability on $[0, 1]$.

Definition 4. *A probability measure θ on $[0, 1]$ is called* stationary *for a transition $K(\cdot, \cdot)$, if*

$$\theta(B) = \int K(x, B)d\theta(x), \qquad \textit{for all interval } B \in [0, 1].$$

More explicitly we assume $K : [0, 1]^2 \to [0, +\infty)$ and $\theta : [0, 1] \to [0, +\infty)$ satisfy the following equations:

$$\int K(x, y)\, dy = 1, \qquad \forall\, x \in [0, 1], \tag{11}$$

$$\int \theta(x)\, K(x, y)\, dxdy = 1, \tag{12}$$

$$\int \theta(x)\, K(x, y)\, dx = \theta(y), \qquad \forall\, y \in [0, 1]. \tag{13}$$

Given the initial probability measure θ and the transition K, as above, one can define a Markov process $\{X_n\}_{n \in \mathbb{N}}$ with state space $[0, 1]$ (see [38] for more details). The measure μ over $[0, 1]^{\mathbb{N}}$ which describes this process is

$$\mu(A_0...A_n \times [0, 1]^{\mathbb{N}}) := \int_{A_0...A_n} \theta(x_0)\, K(x_0, x_1)...K(x_{n-1}, x_n)\, dx_n...dx_0$$

for any cylinder $A_0...A_n \times [0, 1]^{\mathbb{N}}$.

If θ is stationary the Markov Process X_n will be stationary.

Note that θ_β above is stationary for $K_\beta(x, y)$. In this way we can define $\nu_\beta = \theta_\beta K_\beta$ on $[0, 1]^2$.

For instance,

$$\mu_{\beta, A}(\,[a_1, a_2] \times [b_1, b_2] \times [c_1, c_2] \times [0, 1]^{\mathbb{N}}) =$$

$$= \int_{a_1}^{a_2} \int_{b_1}^{b_2} \int_{c_1}^{c_2} \theta_\beta(x_0)\, K_\beta(x_0, x_1)\, K_\beta(x_1, x_2)dx_2\, dx_1\, dx_0. \tag{14}$$

The next result is similar to the one described in [55].

Theorem 16. *Suppose A is a Holder continuous function. Then the probability measure $\mu_{\beta, A}$ defined in (14) is the Gibbs state for the potential βA.*

Proof. We need to show that $\mathcal{L}_{\bar{A}}^*(\mu_{\beta,A}) = \mu_{\beta,A}$, where $\beta\bar{A} = \beta A + \log\psi_\beta - \log\psi_\beta \circ \sigma - \log\lambda_\beta$. Indeed, let $g \in \mathcal{C}$ such that $g(x_0, x_1, ...) = g(x_0, ..., x_k)$, by definition of $\mathcal{L}_{\bar{A}}^*$ we have

$$\int_{\mathcal{B}} g \, d\mathcal{L}_{\bar{A}}^*(\mu_{\beta,A}) = \int_{\mathcal{B}} \mathcal{L}_{\bar{A}}(g) \, d\mu_{\beta,A}$$

$$= \int_{[0,1]^k} \left[\int_{[0,1]} e^{\beta\bar{A}(a,x_0)} g(a, x_0, ..., x_{k-1}) \, da \right] \theta_\beta(x_0) \prod_{j=0}^{k-2} K_\beta(x_j, x_{j+1}) dx_{k-1}...dx_0$$

$$= \int_{[0,1]^k} \left[\int_{[0,1]} e^{\beta A(a,x_0)} \frac{\psi_\beta(a)}{\lambda_\beta \psi_\beta(x_0)} g(a, ..., x_{k-1}) \, da \right] \theta_\beta(x_0) \prod_{j=0}^{k-2} K_\beta(x_j, x_{j+1}) dx_{k-1}...dx_0$$

$$= \int_{[0,1]^{k+1}} e^{\beta A(a,x_0)} \frac{\psi_\beta(a)}{\lambda_\beta} g(a, ..., x_{k-1}) \frac{\bar{\psi}_\beta(x_0)}{\pi_\beta} \prod_{j=0}^{k-2} K_\beta(x_j, x_{j+1}) \, dx_{k-1}...dx_0 \, da$$

$$= \int_{[0,1]^{k+1}} g(a, ..., x_{k-1}) e^{\beta A(a,x_0)} \frac{\bar{\psi}_\beta(x_0)}{\lambda_\beta \bar{\psi}_\beta(a)} \frac{\bar{\psi}_\beta(a) \psi_\beta(a)}{\pi_\beta} \prod_{j=0}^{k-2} K_\beta(x_j, x_{j+1}) \, dx_{k-1}...dx_o \, da$$

$$= \int_{[0,1]^{k+1}} g(a, x_0, ..., x_{k-1}) \theta_\beta(a) K_\beta(a, x_0) \prod_{j=0}^{k-2} K_\beta(x_j, x_{j+1}) \, dx_{k-1}...dx_0 \, da$$

$$= \int_{[0,1]^{k+1}} g(x_0, x_1, ..., x_{k-1}, x_k) \theta_\beta(x_0) \prod_{j=0}^{k-2} K_\beta(x_j, x_{j+1}) K_\beta(x_{k-1}, x_k) \, dx_k...dx_1 \, dx_0.$$

Hence, for any continuous g

$$\int_{\mathcal{B}} g(x_0, ..., x_k) \, d\mathcal{L}_{\bar{A}}^*(\mu_{\beta,A}) = \int_{\mathcal{B}} g(x_0, ..., x_k) d\mu_{\beta,A}.$$

$\square$

The entropy (as defined in section 3) of such probability measure $\mu_{\beta A}$ is

$$h(\mu_{\beta A}) = -\int A(y) \, I_{\{A - \log\lambda_\beta \leq 0\}}(y) \, d\mu_{\beta A}(y) + \log\lambda_\beta.$$

Definition 5. *We denote by $\mathcal{M}_0$ the set of all $\nu = \theta K$ on $[0,1]^2$, where θ is stationary for K.*

Definition 6. *For an absolutely continuous probability measure $\nu \in \mathcal{M}_{[0,1]^2}$, given by a density $\nu(x,y)dxdy$, we denote $S[\nu]$ by*

$$S[\nu] = -\int \nu(x,y) \log\left(\frac{\nu(x,y)}{\int \nu(x,z)dz} \right) dxdy. \tag{15}$$

23

Remark.

The $S[\nu]$ was called "penalized entropy" in [38] and it is a kind of relative entropy with respect to Lebesgue measure. It is different kind of definition of entropy (from the previous one we consider before).

It is easy to see that any $\nu = \theta K \in \mathcal{M}_0$ satisfies

$$S[\theta P] = -\int \theta(x)K(x,y)\log\left(K(x,y)\right)dxdy\,. \tag{16}$$

The value $S[\theta\,K]$ assume negative values.

We can consider now the variational problem

$$P(A) = \max_{\nu=\theta K \in \mathcal{M}_0}\left\{\int \beta A(x,y)\,d\nu + S[\nu]\right\}. \tag{17}$$

This is equivalent to maximize

$$\max_{\nu=\theta K \in \mathcal{M}_0}\left\{\int \beta A(x,y)\theta(x)K(x,y)dxdy - \int \theta(x)K(x,y)\log\left(K(x,y)\right)dxdy\right\}$$

Definition 7. *A probability measure ν in $\mathcal{M}_0$ is called an equilibrium state for A (which depends on two coordinates) if attains the maximal value $P(A)$. The value $P(A)$ is called the pressure (or Free Energy) of A*

We refer the reader to [38] for the proof of the following result.

Proposition 17. *The stationary measure $\nu_\beta = \theta_\beta K_\beta$ defined above maximize*

$$\beta \int A(x,y)\,d\nu + S[\nu],$$

over all stationary $\nu = \theta K \in \mathcal{M}_0$. Also

$$P(A) = \log \lambda_\beta = \int \beta A\,\theta_\beta K_\beta dxdy + S[\theta_\beta K_\beta].$$

When the potential A depends just on two coordinates the equation used in the definition of subaction can be simplified.

Definition 8. *A continuous function $u : [0,1] \to \mathbb{R}$ is called a $[0,1]$- calibrated forward-subaction if, for any $y \in [0,1]$, we have*

$$u(y) = \max_{a\in[0,1]}[A(ay) + u(a) - m(A)]. \tag{18}$$

We refer the reader to [14] for related problems in a different setting. The equation for u above also appears in problems related to the additive eigenvalue [13] [14].

A function u as above can be seen as a function on $x \in [0,1]^{\mathbb{N}}$, where $x = (x_0,x_1,x_2,x_3,...)$, which depends just on the first coordinate x_0. Therefore, a $[0,1]$- calibrated forward-subaction is a also calibrated subaction (in the previous sense). We point out that $[0,1]$- calibrated forward-subactions do exist (see [38]).

An interesting question on the case of selection of measures $\mu_\beta \to \mu_\infty$ is: what happens with the measure of a particular subset D of $\mathcal{B}$ when $T \to 0$ (or, $\beta \to \infty$)? A Large Deviation Principle (see [18] for general references) is true under certain conditions. We refer the reader to [38] for the proof of the result below.

Theorem 18. *If A has only one maximizing probability measure μ_∞ and there exist an unique $[0, 1]$- calibrated forward-subaction V for A, then the following LDP is true: for each cylinder $D = A_0....A_k \times [0, 1]^{\mathbb{N}}$, the following limit exists*

$$\lim_{\beta \to \infty} \frac{1}{\beta} \ln \mu_{\beta A}(D) = - \inf_{\mathbf{x} \in D} I(\mathbf{x}) \, .$$

where $I : [0, 1]^{\mathbb{N}} \to [0, +\infty]$ is a function defined by

$$I(\mathbf{x}) := \sum_{i \geq 0} V(x_{i+1}) - V(x_i) - (A - m(A))(x_i, x_{i+1}) \, .$$

Results about Large deviations in the setting of Thermodynamic Formalism appear in [15] [39].

Definition 9. *We say that $A : [0, 1]^2 \to \mathbb{R}$ satisfies the twist condition, if A is C^2, and*

$$\frac{\partial^2 A}{\partial x \partial y} \neq 0.$$

This property is an open condition under the right topology.

The next theorem (see [38] for a proof) addresses the question of uniqueness when we add a magnetic term $f(x)$ to $A(x, y)$. Related results in a different setting appear in [2] [6]. The above condition for A replaces the convexity of the Lagrangian which is crucial in Aubry-Mather theory [12].

Definition 10. *We will say that a property is* generic *for A, $A \in C^2([0, 1]^2)$, in Mañé's sense, if the property is true for $A + f$, for any f, $f \in C^2([0, 1])$, in a set G which is generic (in Baire sense).*

This concept was initially introduced in the Aubry-Mather setting in [42].

We will show below that under the twist condition the uniqueness of $[0, 1]$-forward backward-subaction is generic in Mañé's sense.

Theorem 19. *Consider the class of all $A : [0, 1]^2 \to \mathbb{R}$ which is C^2 and satisfies the twist condition $\frac{\partial^2 A}{\partial x \partial y} \neq 0$, then there exists a generic set $\mathcal{O}$ in $C^2([0, 1])$ (in Baire sense) such that:*

(a) for each $f \in \mathcal{O}$, $f : [0, 1] \to \mathbb{R}$, for "any" A we have that given $\mu, \tilde{\mu} \in \mathcal{M}_\sigma$ two maximizing measures for $A + f$ (i.e., $m(A + f) = \int (A + f) \, d\mu = \int (A + f) \, d\tilde{\mu}$), then

$$\nu = \tilde{\nu},$$

where ν and $\tilde{\nu}$ are the projections of μ and $\tilde{\mu}$ in the first two coordinates.

(b) for "any" A the $[0, 1]$-calibrated forward-subaction for $A + f$ is unique, for each $f \in \mathcal{O}$ (up to an additive constant).

In the above theorem the potential A is considered the interaction and f the magnetic term. Therefore, it claims, among other things, that for "any" A we have uniqueness of the calibrated subaction (up to an additive constant) for a generic magnetic term f.

The next theorem (see [38] for a proof) addresses the question of the graph property for a probability measure. A related result in the setting of Thermodynamic formalism appears in [41].

Theorem 20. *If $A : [0,1]^2 \to \mathbb{R}$ is C^2, and satisfies the twist condition $\frac{\partial^2 A}{\partial x \partial y} \neq 0$, then, the projected measure ν on $[0,1]^2$ of the maximizing probability measure μ_∞ (on $\mathcal{B}$) has support on a graph.*

The problem we consider above can be seen as a Transshipment Problem (see [38]). For related results see also [24] and [16].

The graph property of a measure is of great importance in Aubry-Mather Theory [12] [21] [43].

5   DLR Gibbs Measures and Transfer Operator

5.1   One-Dimensional Systems and Transfer Operator

Given the potential A we will use the following terminology: Gibbs-TF for A denotes the set of measures usually considered in the Thermodynamical Formalism (as, for example, in [45], or in the first part of this paper) and Gibbs-DLR for A the set of measures constructed as in the Dobrushin-Lanford-Ruelle formulation of Statistical Mechanics, where the Gibbs measures are obtained from Specification Theory point of view, for a complete exposition see [27, 48, 50, 53]. For reasons that will be clarified latter we adopt the notation $\mu^{A,\sigma'}$ for this measures, where σ' is an element of the state space which is called sometimes a boundary condition.

The measures obtained by the the first construction (Section 1) are denoted here by $m = m_A$, and they are defined over the σ-algebra of $\mathcal{B} = (S^1)^{\mathbb{N}}$ generated by the cylinder sets. The second one is usually defined over the σ-algebra of $\mathcal{B}_i = (S^1)^{\mathbb{L}}$ generated by the cylinder sets, where $\mathbb{L}$ is any countable set. In order to show the relation of this two constructions in this paper, we focus on the cases where $\mathbb{L} = \mathbb{Z}$.

We will call M_A the Gibbs-TF-$\mathbb{Z}$ for A, which is, by definition, the natural extension of m_A, the Gibbs-TF for A.

For a large class of potentials (see [27]) we can show that $\mu^{A,\sigma'}$ is independent of the choice of $\sigma' \in (S^1)^{\mathbb{N}}$. Here using a very simple argument we give a proof of this independence using Ruelle operator when one consider free on the left, and a fixed $\sigma' \in (S^1)^{\mathbb{N}}$ boundary conditions. We also show that this unique probability measure constructed using the Gibbs-DLR approach is equals to the measure M_A obtained in the Gibbs-TF-$\mathbb{Z}$ for A. In a forthcoming paper we discus in great generality the equivalence of Gibbs-TF and Gibbs-DLR for one dimensional systems.

5.2 Gibbs-DLR Probability Measures on $(\mathbb{S}^1)^{\mathbb{Z}}$.

For a $\mathcal{B}_i$-measurable function $A : \mathcal{B}_i \to \mathbb{R}$ depending on the two first coordinates, we associated a family $\Phi = (\Phi_\Gamma)_{\Gamma \subset \mathbb{N}}$ of functions from $\mathcal{B}_i$ to $\mathbb{R}$, given by

$$\Phi_\Gamma(x) = \begin{cases} -A(x_n, x_{n+1}), & \text{if } \Gamma = \{n, n+1\}; \\ 0, & \text{otherwise.} \end{cases}$$

We call this family Φ an interaction. For each $n \in \mathbb{N}$ we consider the associated Hamiltonian

$$H^\Phi_{\Lambda_n}(x) = - \sum_{k=-n}^{n-1} A(x_k, x_{k+1}), \tag{19}$$

where $\Lambda_n = [-n, n] \cap \mathbb{Z}$.

The first step to obtain a Gibbs-DLR probability measure for a given $A : \mathcal{B}_i = (\mathbb{S}^1)^{\mathbb{Z}} \to \mathbb{R}$ depending on the two first coordinates, whit boundary condition $\sigma' \in (\mathbb{S}^1)^{\mathbb{N}}$, is to construct a family of probability measures $\mu^{\Phi, \sigma'}_{\Lambda_n}$ over $\mathcal{B}_i$ and then take cluster points in the weak* topology of this family when $n \to \infty$. Note that at least one cluster point exists because of the Banach-Alaoglu Theorem and any element on the set of these cluster points will be called a Gibbs-DLR measure. Once we take the limit when n goes to infinity, the sequence of the sets $\Lambda_n = \{-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n\}$ converges in the set theoretical sense to $\mathbb{Z}$, which allows for these measures to capture information in the past and in the future coordinates.

Fixed a configuration $\sigma' = (\sigma'_0, \sigma'_1, \ldots, \sigma'_n, \ldots) \in (\mathbb{S}^1)^{\mathbb{N}}$, and, a potential A as above, then, we define the Hamiltonian on Λ_n for the potential Φ with σ' right boundary conditions by

$$H^\Phi_{\Lambda_n}(\tau | \sigma') = - \sum_{k=-n}^{n-2} A(\tau_k, \tau_{k+1}) - A(\tau_{n-1}, \sigma'_n).$$

Note that $H^\Phi_{\Lambda_n}(\tau | \sigma')$ can also be considered as a function defined on $[0, 2\pi]^{2n}$, i.e.,

$$H^\Phi_{\Lambda_n}(\tau_{-n}, \ldots, \tau_{n-1} | \sigma') = - \sum_{k=-n}^{n-2} A(\tau_k, \tau_{k+1}) - A(\tau_{n-1}, \sigma'_n).$$

Let $M(\Lambda_n, \sigma') = \{x \in (\mathbb{S}^1)^{\mathbb{Z}} \mid x_i = \sigma'_i , \ \forall \ i \geq n\}$, $d\nu$ the Lebesgue probability measure on $\mathbb{S}^1$ (which we identify with $[0, 2\pi]$) and $d\nu_n$ is the Lebesgue probability measure on $(\mathbb{S}^1)^n$.

The partition function associated to the potential Φ with right boundary condition $\sigma' \in \mathcal{B}$ on the volume Λ_n is defined by

$$
\begin{aligned}
Z^{\Phi, \sigma'}_{\Lambda_n} &:= \int_{M(\Lambda_n, \sigma')} e^{-H^\Phi_{\Lambda_n}(\tau | \sigma')} \, d\nu(\tau) \\
&= \int_{[0, 2\pi]^{2n}} e^{-H^\Phi_{\Lambda_n}(\tau_{-n}, \ldots, \tau_0 \ldots, \tau_{n-1} | \sigma')} \, d\nu(\tau_{-n}) \ldots d\nu(\tau_0) \ldots d\nu(\tau_{n-1}).
\end{aligned}
$$

27

We restrict our attention to potentials Φ for which the partition $Z_{\Lambda_n}^{\Phi,\sigma'}$ is finite for any choice n and σ'. Hence for each n, this defines a probability measure which acts on continuous functions $f : \mathcal{B} \to \mathbb{R}$ (depending on finite coordinates) by

$$\int_{\mathcal{B}_i} f \, d\mu_{\Lambda_n}^{\Phi,\sigma'} = \frac{1}{Z_{\Lambda_n}^{\Phi,\sigma'}} \int_{M(\Lambda_n,\sigma')} f(\tau) \, e^{-H_{\Lambda_n}^{\Phi}(\tau|\sigma')} d\nu(\tau_{-n}) d\nu(\tau_{-n+1}) \dots d\nu(\tau_{n-1}) \,.$$

Note that in this way for any fixed σ' the probability measure $\mu_{\Lambda_n}^{\Phi,\sigma'}$ depends just on A (and, of course, σ'), thus we could also denoted it $\mu_{\Lambda_n}^{A,\sigma'}$. But here we will adopt the Statistical Mechanics notation $\mu_{\Lambda_n}^{\Phi,\sigma'}$ as used in [27] and [50].

For a fixed σ' we are interested in the limit of $\mu_{\Lambda_n}^{\Phi,\sigma'}$, when $n \to \infty$. Any possible cluster point of this sequence will be denoted by $\mu^{A,\sigma'}$ (or, $\mu^{\Phi,\sigma'}$). Any one of these is called a Gibbs state for A with a boundary condition $\sigma' \in \mathcal{B}$ on the right and free on the left.

Given $A : \mathcal{B} \to \mathbb{R}$, by the major theorem of section 1, we know there is a maximal positive eigenvalue $\lambda = \lambda_A$ associated to the eigen-function ψ_A. We also have, for any $\psi : \mathcal{B} \to \mathbb{R}$,

$$\mathcal{L}_A^n \psi(y) = \int_{[0,2\pi]^n} e^{S_n A(\tau y)} \psi(\tau y) \, d\nu_n(\tau) \,. \tag{20}$$

If A depends on two coordinates, then, ψ_A depends on one coordinate (as we get from section 4). Note that for any $\tau \in (\mathbb{S}^1)^{\mathbb{N}}$ we have $\mathcal{L}_A^{2n}(1)\,(\sigma^n(\tau)) = Z_{\Lambda_n}^{\Phi,\tau}$, where σ is the shift on $(\mathbb{S}^1)^{\mathbb{N}}$ and $\mathcal{L}_{\bar{A}}^n 1 = 1$ for any $n \in \mathbb{N}$, where

$$\bar{A} = A + \log \psi_A - \log \psi_A \circ \sigma - \log \lambda_A.$$

Let $\bar{\Phi}$ be the potential defined by $\bar{A}$ and π the natural projection of $(\mathbb{S}^1)^{\mathbb{Z}}$ to $(\mathbb{S}^1)^{\mathbb{N}}$. (analogous to the case for the potential A), we set for any Borelian $C \subset \mathcal{B}$

$$\pi \, \mu_{\Lambda_n}^{\bar{\Phi},\sigma'}(C) = \frac{1}{Z_{\Lambda_n}^{\bar{\Phi},\sigma'}} \int_{M(\Lambda_n,\sigma')} \mathbb{1}_C(\tau) e^{-H_{\Lambda_n}^{\bar{\Phi}}(\tau|\sigma')} d\nu(\tau) \,.$$

We point out that a potential A which depends on two coordinates can be seen as a potential defined either in $(\mathbb{S}^1)^{\mathbb{N}}$, or $(\mathbb{S}^1)^{\mathbb{Z}}$. Another important remark is when the spin variables take values in the close interval $[-1,1]$ these models are known in the literature as continuous Ising model.

Proposition 21. *Consider a fixed $\sigma' \in \mathcal{B} = (\mathbb{S}^1)^{\mathbb{N}}$ Given $A : \mathcal{B}_i \to \mathbb{R}$, which depends on two coordinates, if $\bar{A}$ is its normalized associated potential then for any cluster point $\pi\mu^{\bar{\Phi},\sigma'}$ we have that*

$$m = \pi\mu^{\bar{\Phi},\sigma'},$$

where $m = m_{\bar{A}}$ is the Gibbs-TF measure for $\bar{A}$.

We will show that $\lim_{n\to\infty} \pi\mu_{\Lambda_n}^{\bar{\Phi},\sigma'} = m$, so this limit does not depend on the fixed σ' we choose.

Proof. Consider a given $f : \mathcal{B} \to \mathbb{R}$ which depends on finitely many coordinates, (let's say $r > 0$). Note that

$$H^{\bar{\Phi}}_{\Lambda_n}(\tau | \sigma') = - \sum_{k=-n}^{n-2} \bar{A}(\tau_k, \tau_{k+1}) - \bar{A}(\tau_{n-1}, \sigma'_n),$$

and that $Z^{\bar{\Phi}, \sigma'}_{\Lambda_n} = 1$. Suppose that $n > r$. By definition

$$
\begin{aligned}
\int f \, d\pi \mu^{\bar{\Phi}, \sigma'}_{\Lambda_n} &= \int_{M(\Lambda_n, \sigma')} f(\tau) \, e^{-H^{\bar{\Phi}}_{\Lambda_n}(\tau | \sigma')} d\nu(\tau) \\[2mm]
&= \int_{[0,2\pi]^{2n}} f(\tau_0, ..., \tau_r) \, e^{-H^{\bar{\Phi}}_{\Lambda_n}(\tau_{-n}...\tau_{n-1} | \sigma')} d\nu(\tau_{-n})...d\nu(\tau_{n-1}) \\[2mm]
&= \int_{[0,2\pi]^n} f(\tau_0, ..., \tau_r) e^{\sum_{k=0}^{n-2} \bar{A}(\tau_k, \tau_{k+1}) + \bar{A}(\tau_{n-1}, \sigma'_n)} \times \\[2mm]
&\quad \times \left(\int_{[0,2\pi]^n} e^{\sum_{k=-n}^{-1} \bar{A}(\tau_k, \tau_{k+1})} d\nu(\tau_{-n})...d\nu(\tau_{-1}) \right) d\nu(\tau_0) ... d\nu(\tau_{n-1}) \\[2mm]
&= \int_{[0,2\pi]^n} f(\tau_0, ..., \tau_r) e^{\sum_{k=0}^{n-2} \bar{A}(\tau_k, \tau_{k+1}) + \bar{A}(\tau_{n-1}, \sigma'_n)} d\nu(\tau_0) ... d\nu(\tau_{n-1}).
\end{aligned}
$$

where in the last equation we used n times $\int_{[0,2\pi]} e^{\bar{A}(x,y)} d\nu(x) = 1$.

In this way,

$$\int f \, d\pi \mu^{\bar{\Phi}, \sigma'}_{\Lambda_n} = \mathcal{L}^n_{\bar{A}}(f)(\sigma^n(\sigma')).$$

Is is known from section 1 that $\mathcal{L}^n_{\bar{A}}(f)$ converges uniformly to $\int f \, dm$, as n goes to infinity, where m is Gibbs-TF for A or $(\bar{A})$. As the convergence of $\mathcal{L}^n_{\bar{A}}(f)$, when $n \to \infty$, is uniform, then $\lim_{n \to \infty} \pi \mu^{\bar{\Phi}, \sigma'}_{\Lambda_n} = m$

$\square$

Corollary 22. *For any $\sigma' \in (\mathbb{S}^1)^{\mathbb{N}}$, and, any f which depends on finitely many coordinates*

$$\frac{\int_{M(\Lambda_n, \sigma')} f(\tau) \, e^{-H^{\bar{\Phi}}_{\Lambda_n}(\tau | \sigma')} d\nu(\tau_{-n}) d\nu(\tau_{-n+1}) ... d\nu(\tau_{n-1})}{\int_{(\mathbb{S}^1)^{\Lambda_n}} f(\tau) \, e^{-H^{\bar{\Phi}}_{\Lambda_n}(\tau)} d\nu(\tau_{-n}) d\nu(\tau_{-n+1}) ... d\nu(\tau_{n-1}) d\nu(\tau_n)} \to 1,$$

when $n \to \infty$.

Proof. This follows easily from the above because the convergence of $\mathcal{L}^n_{\bar{A}}(f)$ is uniform.

$\square$

Proposition 23. *Suppose $\sigma' \in (\mathbb{S}^1)^{\mathbb{N}}$. Given $A : \mathcal{B}_i \to \mathbb{R}$, which depends on two coordinates and, a coboundary $h : \mathcal{B}_i \to \mathbb{R}$, which depends on one coordinate (the 0 coordinate), and, such that*

$$\bar{A} = A + h - h \circ \hat{\sigma} + \log \lambda,$$

where $\hat{\sigma}$ is the shift on $\mathcal{B}_i$, then

$$\pi(\mu^{\Phi,\sigma'}) = \pi(\mu^{\bar{\Phi},\sigma'}).$$

Proof. Consider a function $f : \mathcal{B} \to \mathbb{R}$ which depends on finite coordinates $f(\tau_0, \tau_1, .., \tau_k)$, $k > 0$. We have first that

$$H_{\Lambda_n}^{\bar{\Phi}}(\tau|\sigma') \;=\; -\bar{A}(\tau_{-n}, \tau_{-n+1}) - \sum_{k=-n+1}^{n-2} \bar{A}(\tau_k, \tau_{k+1}) - \bar{A}(\tau_{n-1}, \sigma_n')$$

$$=\; H_{\Lambda_n}^{\Phi}(\tau|\sigma') + h(\sigma_n') - h(\tau_{-n}) - 2n \log \lambda.$$

Hence

$$-H_{\Lambda_n}^{\Phi}(\tau|\sigma') = -H_{\Lambda_n}^{\bar{\Phi}}(\tau|\sigma') + h(\sigma_n') - h(\tau_{-n}) - 2n \log \lambda.$$

Therefore

$$\int_{M(\Lambda_n,\sigma')} f(\tau) \, e^{-H_{\Lambda_n}^{\Phi}(\tau|\sigma')} \, d\nu(\tau) =$$

$$\lambda^{-2n} \, e^{h(\sigma_n')} \int_{M(\Lambda_n,\sigma')} e^{-h(\tau_{-n})} f(\tau_0, .., \tau_k) \, e^{-H_{\Lambda_n}^{\bar{\Phi}}(\tau|\sigma')} \, d\nu(\tau_{-n}) \ldots d\nu(\tau_{n-1}),$$

by taking $f = 1$ we have

$$Z_{\Lambda_n}^{\Phi,\sigma'} = \int_{M(\Lambda_n,\sigma')} e^{-H_{\Lambda_n}^{\Phi}(\tau|\sigma')} \, d\nu(\tau) =$$

$$\lambda^{-2n} \, e^{h(\sigma_n')} \int_{M(\Lambda_n,\sigma')} e^{-h(\tau_{-n})} \, e^{-H_{\Lambda_n}^{\bar{\Phi}}(\tau|\sigma')} \, d\nu(\tau_{-n}) \ldots d\nu(\tau_{n-1}).$$

$$= \lambda^{-2n} \, e^{h(\sigma_n')} \mathcal{L}_{\bar{A}}^{2n}(e^{-h})(\sigma^n(\sigma')).$$

We already shown in the previous sections that

$$\mathcal{L}_{\bar{A}}^{2n}(e^{-h})(\sigma^n(\sigma')) \to \int e^{-h} \, dm_{\bar{A}},$$

uniformly in n. Therefore, $Z_{\Lambda_n}^{\Phi,\sigma'} \sim \lambda^{-2n} e^{h(\sigma_n')} \int e^{-h} \, dm_{\bar{A}}$.

We also have

$$\int_{M(\Lambda_n,\sigma')} e^{-h(\tau_{-n})} f(\tau_0, \tau_1, .., \tau_k) \, e^{-H_{\Lambda_n}^{\bar{\Phi}}(\tau|\sigma')} \, d\nu(\tau) =$$

$$\int_{[0,2\pi]^n} f(\tau) e^{\sum_{k=0}^{n-2} \bar{A}(\tau_k,\tau_{k+1}) + \bar{A}(\tau_{n-1},\sigma_n')} \; \times$$

30

$$\times \left(\int_{[0,2\pi]^n} e^{-h(\tau_{-n})} e^{\sum_{k=-n}^{-1} \bar{A}(\tau_k, \tau_{k+1})} \prod_{i=-n}^{-1} d\nu(\tau_i) \right) \prod_{k=0}^{n-1} d\nu(\tau_k) =$$

$$\int_{[0,2\pi]^n} f(\tau) e^{\sum_{k=0}^{n-2} \bar{A}(\tau_k, \tau_{k+1}) + \bar{A}(\tau_{n-1}, \sigma'_n)} (\mathcal{L}_{\bar{A}}^n(e^{-h})(\tau)) \prod_{k=0}^{n-1} d\nu(\tau_k) =$$

$$\int_{[0,2\pi]^n} f(\tau) e^{\sum_{k=0}^{n-2} \bar{A}(\tau_k, \tau_{k+1}) + \bar{A}(\tau_{n-1}, \sigma'_n)} \left(\mathcal{L}_{\bar{A}}^n(e^{-h})(\tau) - \int e^{-h} dm_{\bar{A}} \right) \prod_{k=0}^{n-1} d\nu(\tau_k) +$$

$$+ \int_{[0,2\pi]^n} f(\tau) e^{\sum_{k=0}^{n-2} \bar{A}(\tau_k, \tau_{k+1}) + \bar{A}(\tau_{n-1}, \sigma'_n)} \left(\int e^{-h} dm_{\bar{A}} \right) \prod_{k=0}^{n-1} d\nu(\tau_k)$$

$$\to \int f dm_{\bar{A}} \int e^{-h} dm_{\bar{A}}.$$

where in the convergence we used the fact that, given any $\epsilon > 0$, there exists N_ϵ such that, for $n > N_\epsilon$, we have

$$\left| \int_{[0,2\pi]^n} f(\tau) e^{\sum_{k=0}^{n-2} \bar{A}(\tau_k, \tau_{k+1}) + \bar{A}(\tau_{n-1}, \sigma'_n)} \left(\mathcal{L}_{\bar{A}}^n(e^{-h})(\tau) - \int e^{-h} dm_{\bar{A}} \right) \prod_{k=0}^{n-1} d\nu(\tau_k) \right| <$$

$$< \epsilon \int_{[0,2\pi]^n} f(\tau) e^{\sum_{k=0}^{n-2} \bar{A}(\tau_k, \tau_{k+1}) + \bar{A}(\tau_{n-1}, \sigma'_n)} \prod_{k=0}^{n-1} d\nu(\tau_k) =$$

$$= \epsilon \mathcal{L}_{\bar{A}}^n(f)(\sigma^n(\sigma')) < 2\epsilon \int f dm_{\bar{A}},$$

which means that the first integral vanishes when $n \to \infty$, while the second integral is

$$\int_{[0,2\pi]^n} f(\tau) e^{\sum_{k=0}^{n-2} \bar{A}(\tau_k, \tau_{k+1}) + \bar{A}(\tau_{n-1}, \sigma'_n)} \left(\int e^{-h} dm_{\bar{A}} \right) \prod_{k=0}^{n-1} d\nu(\tau_k) =$$

$$= \mathcal{L}_{\bar{A}}^n(f)(\sigma^n(\sigma')) \int e^{-h} dm_{\bar{A}} \to \int f dm_{\bar{A}} \int e^{-h} dm_{\bar{A}}.$$

Finally,

$$\frac{\int_{M(\Lambda_n, \sigma')} f(\tau) e^{-H_{\Lambda_n}^{\Phi}(\tau|\sigma')} d\nu(\tau)}{Z_{\Lambda_n}^{\Phi, \sigma'}}$$

$$= \frac{\lambda^{-2n} e^{h(\sigma'_n)} \int_{M(\Lambda_n, \sigma')} e^{-h(\tau_{-n})} f(\tau) e^{-H_{\Lambda_n}^{\bar{\Phi}}(\tau|\sigma')} d\nu(\tau)}{Z_{\Lambda_n}^{\Phi, \sigma'}}$$

$$\sim \frac{\int_{M(\Lambda_n, \sigma')} e^{-h(\tau_{-n})} f(\tau) e^{-H_{\Lambda_n}^{\bar{\Phi}}(\tau|\sigma')} d\nu(\tau)}{\int e^{-h} dm_{\bar{A}}} \to \int f dm_{\bar{A}}$$

Therefore, $\pi \mu^{\Phi, \sigma'} = \pi \mu^{\bar{\Phi}, \sigma'}$.

$\square$

31

Corollary 24. *Consider a general $\sigma' \in \mathcal{B}$. Given $A : \mathcal{B} \to \mathbb{R}$, then,*

$$m_A = \pi \mu^{\Phi,\sigma'},$$

where $m = m_A$ is the Gibbs-TF for A.

Proof. It follows from $\bar{A} = A + \log \psi_A - \log \psi_A \circ \sigma - \log \lambda_A$.

$\square$

According to [27] Part III page 289, for any σ' the probability measure $\mu^{A,\Phi,\sigma'}$ is invariant for $\hat{\sigma}$ acting on $(\mathbb{S}^1)^{\mathbb{Z}}$.

By definition, the Gibbs-FT-Z state M_A on $(\mathbb{S}^1)^{\mathbb{Z}}$, is the natural extension of m_A, and, it is also invariant for $\hat{\sigma}$ acting on $(\mathbb{S}^1)^{\mathbb{Z}}$.

Proposition 25. *Suppose $A : (\mathbb{S}^1)^{\mathbb{Z}} \to \mathbb{R}$ depends on two coordinates, and, consider $\sigma' \in \mathcal{B}$, then*
$$\mu^{\Phi,\sigma'} = M_A.$$

Proof. $\mu^{\Phi,\sigma'}$ and M_A are both the natural extension of m_A.

$\square$

Proposition 26. *Suppose $A : (\mathbb{S}^1)^{\mathbb{Z}} \to \mathbb{R}$ depends on two coordinates, and, consider $\sigma', \sigma'' \in \mathcal{B}$, then*
$$\mu^{\Phi,\sigma'} = \mu^{\Phi,\sigma''}.$$

Proof. $\mu^{\Phi,\sigma'}$ and $\mu^{\Phi,\sigma''}$ are both the natural extension of m_A.

$\square$

The final conclusion is that, if the potential depends on two coordinates, then the Gibbs probability measure on $(\mathbb{S}^1)^{\mathbb{Z}}$ in both settings, Thermodynamic Formalism and Statistical Mechanics via a boundary condition σ' on the right side, coincide.

Now we will analyze the free-boundary case. Remember that

$$H^{\Phi}_{\Lambda_n}(\tau) = - \sum_{k=-n}^{n-1} A(\tau_k, \tau_{k+1}).$$

We are going to define the Gibbs probability measure in the sense of Statistical Mechanics with free boundary condition on the left and on the right. For a given $n > 0$,

$$Z^{\Phi}_{\Lambda_n} = \int_{(\mathbb{S}^1)^{\Lambda_n}} e^{-H^{\Phi}_{\Lambda_n}(\tau)} \, d\nu(\tau_{-n}) d\nu(\tau_{-n+1}) \dots d\nu(\tau_{n-1}) d\nu(\tau_n)$$

will be the partition function which corresponds to the case of free a boundary condition on the right and on the left.

For each n, this defines a probability measure which acts on continuous functions f (depending on finite coordinates) by

$$\int f \, d \, \mu_{\Lambda_n}^{\Phi} \;=\; \frac{1}{Z_{\Lambda_n}^{\Phi}} \int_{(\mathbb{S}^1)^{\Lambda_n}} f(\tau) \, e^{-H_{A,\Lambda_n}^{\Phi}(\tau)} d\nu(\tau_{-n}) d\nu(\tau_{-n+1}) \ldots d\nu(\tau_{n-1}) \, d\nu(\tau_n).$$

Any weak limit of subsequences of $\mu_{\Lambda_n}^{\Phi}$ will be called a Gibbs state for A with a free boundary condition on the right and on the left.

It follows from Corollary 1 above that any Gibbs state for A with a free boundary condition on the right and on the left is equal to M_A.

The result we will analyze in the next section will be the case of a free boundary condition on the right and on the left.

6 An example by A. C. D. van Enter and W. M. Ruszel where there is no selection

In this section we will consider A depending on its first neighbors, and having the form $A(x) = A(x_0, x_1) = U(x_0 - x_1)$.

We want to show a particular example (introduced by [20]), where the potential is not continuous and is of the form: $\tilde{U} : [0, 2\pi] \to \mathbb{R}$ is a function such that $\tilde{U}|_{[a_n, b_n)}$, is constant for each n and equal to c_n, where $[a_n, b_n)$, $n \in \mathbb{N}$ is a partition of $[a, b]$.

We will show that for each positive β we can also consider an extension of Gibbs-TF, say $\mu_{\beta, \tilde{U}}$, over $\mathcal{B}$ and also that this measure coincides with the Gibbs-DLR for this potential $\tilde{U}$. In [20] the authors have shown that there is no selection of the family $\mu_{\beta, \tilde{U}}$ when $\beta \to \infty$.

We will present here all the details of the proof of this non-trivial result.

Basically, we will show that $\int I_B \, d\mu_{\beta, \tilde{U}}$ does not converge when $\beta \to \infty$, for a set B which depends just on the coordinates (x_0, x_1). Therefore, this is also the same as to say that $\int I_B \, d\hat{\mu}_{\beta, \tilde{U}}$ does not converge (see Remark 4 just before proposition 6).

The main result of this section is theorem 32, which is a consequence of corollary 30 and lemma 31. Subsection 6.1 shows that results of previous sections are still valid even if the potential A belongs to certain classes of non-continuous potentials including the potential of [20].

6.1 Gibbs Measures for Non-continuous Potentials and DLR formulation of Statistical Mechanics

So far we have defined Gibbs Measures for Holder continuous potentials in sections 1 (general case) and 4 (nearest neighbors interaction, i.e. potential depending on two coordinates). In the section 4 we gave an alternative definition based on transition kernels associated to a certain potential (or Hamiltonian) A, and proved that this definition is equivalent to the one of section 1.

We will now show that our definition coincides with the usual one in Statistical Mechanics, in the case of a certain special non-continuous potential depending on two coordinates. We assume, among other things, the form $\tilde{A}(x) = \tilde{A}(x_0, x_1) = \tilde{U}(x_0 - x_1)$, where $\tilde{U} : \mathbb{S}^1 \to \mathbb{R}$ is a bounded L^1 function, which is pointwise approximated by Holder functions U_n. This case will cover the important example to be described later. A potential of this form is called symmetric.

First we will show that the main results of Section 4 are true for this potential $\tilde{A}(x) = \tilde{U}(x_0 - x_1)$, which is no longer continuous.

Using the notation described in section 4, let $\mathcal{L}_{\beta\tilde{U}}, \bar{\mathcal{L}}_{\beta\tilde{U}} : C([0, 2\pi]) \to C([0, 2\pi])$ be given by

$$\mathcal{L}_{\beta\tilde{U}}\psi(y) = \frac{1}{2\pi} \int_0^{2\pi} e^{\beta\tilde{U}(x-y)} \psi(x)dx, \tag{21}$$

$$\bar{\mathcal{L}}_{\beta\tilde{U}}\psi(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{\beta\tilde{U}(x-y)} \psi(y)dy. \tag{22}$$

for any $y \in [0, 2\pi]$.

In order to simplify the notation we denote $\mathcal{L}_\beta$ instead of $\mathcal{L}_{\beta\tilde{U}}$.

Lemma 27. *The operators $\mathcal{L}_\beta$ and $\bar{\mathcal{L}}_\beta$ preserve the set of continuous functions in $[0, 2\pi]$, sending continuous functions to uniformly continuous functions. Moreover, a bounded function is mapped to an uniformly continuous one.*

The fact that continuous functions are preserved implies the compactness of the operator, as we can see in pages 43 and 47 of [11].

Proof. Consider a fixed β and the operator $\mathcal{L}_\beta$. Let f be a continuous function.

Fix $\epsilon > 0$. Let A_c be a continuous function such that

$$\|\tilde{A} - A_c\|_{L^1} < \frac{\epsilon}{4\|f\|_{C^0}} .$$

Here we use the L_1 norm on the functions defined on the one-dimensional set $[0, 2\pi]$. Such a function exists because continuous functions are dense in $L_p[0, 2\pi]$ for $p \geq 1$.

Let $K_c(x, y) = A_c(x - y)$. We have $\tilde{A} = A_c + (\tilde{A} - A_c)$

Moreover, let $\delta > 0$ be such that

$$|A_c(z) - A_c(w)| < \frac{\epsilon}{2\|f\|_{C^0}}$$

if $|z - w| < \delta$.

Suppose $|y_1 - y_2| < \delta$. Then we have

$$|\mathcal{L}(f)(y_1) - \mathcal{L}(f)(y_2)| = \left| \int K(x, y_1)f(x)dx - \int K(x, y_2)f(x)dx \right|$$

$$\leq \int |A_c(x - y_1) - A_c(x - y_2)| \, |f(x)|dx +$$

$$+ \int |(\tilde{A} - A_c)(x - y_1)| |f(x)| dx + \int |(\tilde{A} - A_c)(x - y_2)| |f(x)| dx < \epsilon \,.$$

$\square$

The proof of the next theorem is a small modification of the proof of Theorem 3 of [38]

Theorem 28. *The operators $\mathcal{L}_\beta$ and $\bar{\mathcal{L}}_\beta$ have the same positive maximal eigenvalue λ_β, which is simple and isolated. The eigenfunctions associated to these operators, say ψ_β and $\bar{\psi}_\beta$, are positive functions.*

Proof. We can see that $\mathcal{L}_\beta$ is a compact operator, because Lemma 27 shows that the image of the unity closed ball of $C([0,1])$ under $\mathcal{L}_\beta$ is an equicontinuous family in $C([0,1])$. Thus, we can use Arzelà-Ascoli theorem to prove the compactness of $\mathcal{L}_\beta$ (see also Chapter IV, section 1 of [49]).

The spectrum of a compact operator contains a sequence of eigenvalues that converges to zero, possibly contains zero. This implies that any non-zero eigenvalue of $\mathcal{L}_\beta$ is isolated (i.e. there is no sequence in the spectrum of $\mathcal{L}_\beta$ which converges to a non-zero eigenvalue).

The definition of $\mathcal{L}_\beta$ now shows that $\mathcal{L}_\beta$ preserves the cone of positive functions in $C([0,1])$, sending a point in this cone to the interior of the cone. This means that $\mathcal{L}_\beta$ is a positive operator.

The Krein-Ruthman theorem (Theorem 19.3 of [17]) implies that there exists a positive eigenvalue λ_β, which is maximal (i.e. if $\lambda \neq \lambda_\beta$ is in the spectrum of $\mathcal{L}_\beta$ then $\lambda_\beta > |\lambda|$.) and simple (i.e. the eigenspace associated to λ_β is one-dimensional). Moreover λ_β is associated to a positive eigenfunction ψ_β.

If we proceed in the same way as in [38], we obtain the same conclusions about the operator $\bar{\mathcal{L}}_\beta$, and we get the respective eigenvalue $\bar{\lambda}_\beta$ and eigenfunction $\bar{\psi}_\beta$.

In order to prove that $\bar{\lambda}_\beta = \lambda_\beta$, we use the positivity of ψ_β and $\bar{\psi}_\beta$ and the fact that $\bar{\mathcal{L}}_\beta$ is the adjoint of $\mathcal{L}_\beta$. (Here we see that our operators can be, in fact, defined in the Hilbert space $L^2([0,1])$, which contains $C([0,1])$). We have $< \psi_\beta, \bar{\psi}_\beta > = \int \psi_\beta(x) \bar{\psi}_\beta(x) dx > 0$, and

$$\lambda_\beta < \psi_\beta, \bar{\psi}_\beta > = < \mathcal{L}_\beta \psi_\beta, \bar{\psi}_\beta > = < \psi_\beta, \bar{\mathcal{L}}_\beta \bar{\psi}_\beta > = \bar{\lambda}_\beta < \psi_\beta, \bar{\psi}_\beta > \,.$$

$\square$

By the periodicity of $\tilde{U}$, $\mathcal{L}_{\beta\tilde{U}} \psi(1)$ and $\bar{\mathcal{L}}_{\beta\tilde{U}}(1)$ are independent of x. Therefore $\psi_{\beta,\tilde{U}}(x) = \bar{\psi}_{\beta,\tilde{U}}(x) = 1$ are the eigenfunctions associated to the maximal eigenvalue $\lambda_{\beta,\tilde{U}}$.

It is easy to see that

$$\lambda_{\beta,\tilde{U}} = \frac{1}{2\pi} \int_0^{2\pi} e^{\beta\tilde{U}(x-y)} \, dy \,.$$

In the notation section 4, $\theta_{\beta,\tilde{U}}(x) = 1$ and the transition Kernel is given by

$$K_{\beta,\tilde{U}}(x,y) := \frac{e^{\beta\tilde{U}(x-y)}}{\lambda_\beta} \,.$$

For instance, for any cylinder

$$\mu_{\beta,\tilde{U}}(A_0...A_k) = \int_{A_0...A_k} \frac{e^{\beta \sum_{i=0}^{k-1} \tilde{U}(x_i - x_{i+1})}}{\lambda_\beta^k} \, dx_k \, ... \, dx_0.$$

This measure does not came from a Hölder potential, but we can approximate this measure by Gibbs-TF measures associated to Hölders potentials, as we will see next.

Let us now analyze the case $A(x) = A(x_0, x_1) = U(x_0 - x_1)$ where $U : \mathbb{R} \to \mathbb{R}$ is a Holder continuous function 2π-periodic. By the same arguments used above, it is easy to see that $\psi_{\beta,U}(x) = \bar{\psi}_{\beta,U}(x) = 1$ are the the eigenfunctions of the operators $\mathcal{L}_{\beta,U}, \bar{\mathcal{L}}_{\beta,U}$ associated to the maximal eigenvalue $\lambda_{\beta,U}$ (see section 4), where

$$\lambda_{\beta,U} = \frac{1}{2\pi} \int_0^{2\pi} e^{\beta U(x-y)} \, dy \,.$$

As in section 4, $\theta_{\beta,U}(x) = 1$ and the transition Kernel is given by

$$K_{\beta,U}(x,y) := \frac{e^{\beta U(x-y)}}{\lambda_\beta} \,.$$

Hence, for any cylinder

$$\mu_{\beta,U}(A_0...A_k) = \int_{A_0...A_k} \frac{e^{\beta \sum_{i=0}^{k-1} U(x_i - x_{i+1})}}{\lambda_\beta^k} \, dx_k \, ... \, dx_0.$$

By theorem 16 we see that $\mu_{\beta,U} = m_U$, the Gibbs-TF for U.

Let now $\tilde{U}$ be a L^1 potential such that there exists an uniformly bounded sequence of Holder continuous potentials U_n converging point wise to $\tilde{U}$.

By the Dominated Convergence Theorem, we have that

$$\lambda_{\beta,U_n} = \frac{1}{2\pi} \int_0^{2\pi} e^{\beta U_n(x-y)} \, dy \to \frac{1}{2\pi} \int_0^{2\pi} e^{\beta \tilde{U}(x-y)} = \lambda_{\beta,\tilde{U}} \,,$$

as $k \to \infty$, and also for any cylinder $A_0...A_k$, we have

$$\mu_{\beta,U_n}(A_0...A_k) \to \int_{A_0...A_k} \frac{e^{\beta \sum_{i=0}^{k-1} \tilde{U}(x_i - x_{i+1})}}{\lambda_\beta^k} \, dx_k \, ... \, dx_0 = \mu_{\beta,\tilde{U}}(A_0...A_k)$$

as $k \to \infty$.

Note that the measure $\mu_{\beta,\tilde{U}}$ coincides with the Gibbs-DLR measure of statistical mechanics in the special case of nearest neighbors interaction of the kind $\tilde{A}(x) = \tilde{A}(x_0, x_1) = \tilde{U}(x_0 - x_1)$ as can be seen, for example, in [27]. We also remark that $\frac{1}{\lambda_\beta^k}$ is the partition function of DLR formulations of statistical mechanics.

6.2 One-dimensional Systems With Symmetric Potentials

To explain the no selection measure theorem we will use the formalism introduced in last section. Here we take $\mathbb{L} = \mathbb{Z}$ which is the origin of the term "one-dimensional" in the title of this section. We assign for each $i \in \mathbb{Z}$ the measure space $(\mathbb{S}^1, \mathscr{B}, \nu)$, where ν is the Lebesgue probability measure on the circle. For each $n \in \mathbb{N}$ we denote by $\Lambda_n =: [-n, n] \cap \mathbb{Z}$. We will use free conditions on the left and on the right side.

For convenience, we use the natural measure isomorphism between the Bernoulli spaces $(\mathbb{S}^1)^{\mathbb{Z}}$ and $[0, 2\pi)^{\mathbb{Z}}$ to define the Hamiltonian we introduced before in $(\mathbb{S}^1)^{\mathbb{Z}}$. Let $\Phi = (\Phi_\Gamma)_{\Gamma \subset \mathcal{L}}$ be a family of functions on $[0, 2\pi)^{\mathbb{Z}}$, such that

$$
\Phi_\Gamma(\theta) = \begin{cases} U(\theta_k - \theta_{k+1}), & \text{if } \Gamma = \{k, k+1\}; \\ 0, & \text{otherwise.} \end{cases}
$$

where U is a potential defined by the 2π-periodic extension of

$$
\tilde{U}(x) = \sum_{j=1}^{\infty} c_j \mathbb{1}_{A_j}(x),
$$

and $\{A_j\}_{j \geq 1}$ is a partition of $[0, 2\pi)$ given by intervals of the form $A_j = [a_j, b_j)$.

Using the isomorphism and the family Φ mentioned above, the Hamiltonian in the finite volume Λ_n, with boundary condition x', will be given by the following expression, if $x = (\theta_k)_{k \in \mathbb{Z}}$

$$
H_{\Lambda_n}(x_{\Lambda_n} x'_{\Lambda_n^c}) = -\sum_{k=-n}^{n-1} U(\theta_k - \theta_{k+1}) - U(\theta_{-n} - \theta'_{-n-1}) - U(\theta_n - \theta'_{n+1}). \tag{23}
$$

The family Φ we are considering is associated to a potential A which depends only on the nearest-neighbors and given by $A(x, y) = U(x - y)$. We can prove (see [48]) that for each fixed $\beta \in (0, +\infty)$, the set $\mathcal{G}_{\beta, \Phi}$ is a singleton set and its unique measure denoted by $\mu_{\beta A}$ is given by

$$
\mu_{\beta A} = w - \lim_{\Lambda_n \nearrow \mathbb{N}} \mu_{\Lambda_n}^{\beta, \Phi},
$$

where for all $n \in \mathbb{N}$ and $E \in \mathcal{B}_i$

$$
\mu_{\Lambda_n}^{\beta, \Phi}(E) = \frac{1}{Z_{\Lambda_n}^{\beta, \Phi}} \int_{(\mathbb{S}^1)^{\Lambda_n}} \mathbb{1}_{\pi_{\Lambda_n}(E)}(\theta) \exp\left(\beta \sum_{k=-n}^{n-1} U(\theta_k - \theta_{k+1})\right) \prod_{k=-n}^{n} d\nu(\theta_k). \tag{24}
$$

From now on we call $\mu_{\Lambda_n}^{\beta, \Phi}$ the Gibbs measure in the volume Λ_n for the Hamiltonian (23) at inverse temperature β.

We are using above free boundary conditions on the left and on the right side.

We will consider here a real parameter β, wich means the inverse of the temperature, and the Gibbs probability measure $\hat{\mu}_{\beta A}$ over $\mathcal{B}_i = (\mathbb{S}^1)^{\mathbb{Z}}$ (see the considerations just above proposition 6).

Note that if U has a unique maximum at $y = 0 \in \mathbb{S}^1$, then the support of any maximizing probability measure μ_∞ for $A(x,y) = U(x - y)$ is always contained in the set

$$\mathcal{K} = \{x = (...x_{-2}, x_{-1}, x_0, x_1, x_2, ...) \, : \, x_i = c \in \mathbb{S}^1, \, \forall i \in \mathbb{Z}\} \subset \mathcal{B}_i.$$

All points in $\mathcal{K}$ are fixed points for $\hat{\sigma}$. The above set $\mathcal{K}$ can be indexed by $c \in \mathbb{S}^1$. Each fixed point x in this set can be denoted by x_c, where $c \in \mathbb{S}^1$. The corresponding maximizing probability measure for A over $(\mathbb{S}^1)^{\mathbb{Z}}$ is δ_{x_c}.

Given any probability measure P over $\mathbb{S}^1$, we can consider the probability measure ν over $(\mathbb{S}^1)^{\mathbb{Z}}$ given by $\nu = \int \delta_{x_c} dP(c)$. The general maximizing probability measure for A is of this form.

Suppose now that U has two strict maximals at $y = 0 \in \mathbb{S}^1$ and at $y = \pi$. In this case, the support of any maximizing probability measure for $A(x,y) = U(x - y)$ is always contained in the set $\mathcal{K} = \mathcal{K}_1 \cup \mathcal{K}_2$, where

$$\mathcal{K}_1 = \{x = (...x_{-2}, x_{-1}, x_0, x_1, x_2, ...) \, : \, x_i = c \in \mathbb{S}^1, \, \forall i \in \mathbb{Z}\} \subset \mathcal{B}_i,$$

and

$$\mathcal{K}_2 = \{x = (...x_{-2}, x_{-1}, x_0, x_1, x_2, ...) \, : \, x_{i+1} - x_i = \pi \in \mathbb{S}^1, \, \forall i \in \mathbb{Z}\} \subset \mathcal{B}_i.$$

The set $\mathcal{K}_1$ is called the set of ferromagnetic states, and, the set $\mathcal{K}_2$ is called the set of anti-ferromagnetic states. The points in $\mathcal{K}_2$ have $\hat{\sigma}$-period equal to two. A similar result to the above is true for the general maximizing probability measure for A.

Now we will state proposition 29 and its corollary 30, which, together with lemma 31, will be used to prove the main result of this section, the non-selection theorem 32.

Proposition 29. *Let $\mu_{\Lambda_n}^{\beta,\Phi}$ be the Gibbs measure in the volume Λ_n, defined by (24). For any fixed $j \in \mathbb{N}$ and $k \in \{-n, \ldots, n-1\}$, if*

$$B_{k,j} = \{(\theta_{-n}, \ldots, \theta_n) \in (0, 2\pi]^{2n+1} : \theta_k - \theta_{k+1} \in A_j\},$$

then

$$\mu_{\Lambda_n}^{\beta,\Phi}(B_{k,j}) = \frac{1}{Z(\beta)} \nu(A_j) e^{\beta c_j},$$

where

$$Z(\beta) = \frac{1}{2\pi} \int_{(0,2\pi]} e^{\beta U(x)} dx,$$

and $\nu(A_j)$ is the Lebesgue probability measure of A_j.

In fact, to prove theorem 32, we will only need to consider the Borel sets $B_j = \{\theta_0 - \theta_1 \in A_j\} \subset \mathcal{B}_i$, $j \in \mathbb{N}$, because we are interested in estimate $\mu_{\beta A}(B_j) = \int I_{B_j} d\mu_{\beta A}$, for each j, when $\beta \to \infty$.

To state corollary 30 we will consider the potential introduced in [20].

Corollary 30. *Let $\varepsilon > 0$. Consider the special case where*

$$\tilde{U}(x) = \sum_{i=1}^{\infty} \frac{3}{2^{2i+1}} \mathbb{1}_{I_{2i}}(x) + \sum_{i=1}^{\infty} \frac{3}{2^{2i+2}} \mathbb{1}_{I_{2i+1}}(x - \pi) + \frac{1}{4} \mathbb{1}_{I_1}(x - \pi),$$

where $I_i = [-\frac{\varepsilon^{3^i}}{2}, \frac{\varepsilon^{3^i}}{2}]$. For each $j \in \mathbb{N}$ we define the ring A_j as follows. If j is even, then $A_j = A_{2i} = I_{2i} \backslash I_{2i+2}$, and if j is odd then $A_j = A_{2i+1} = I_{2i+1} \backslash I_{2i+3} + \pi$. For any fixed $j \in \mathbb{N}$ and $k \in \{-n, \dots, n-1\}$, we have

$$\mu_{\beta A}(\{\theta_k - \theta_{k+1} \in A_j\}) = \mu_{\Lambda_n}^{\beta; \Phi}(\{\theta_k - \theta_{k+1} \in A_j\}) = \frac{1}{Z(\beta)} \nu(A_j) \exp\left(\frac{\beta}{2} - \frac{\beta}{2^{j+1}}\right),$$

where

$$Z(\beta) = \frac{1}{2\pi} \int_{(0, 2\pi]} e^{\beta U(x)} dx$$

and

$$\nu(A_j) = \varepsilon^{3^j} - \varepsilon^{3^{j+2}}.$$

In particular,

$$
\begin{aligned}
\mu_{\beta A}(\{\theta_0 - \theta_1 \in A_j\}) &= \frac{1}{Z(\beta)} \nu(A_j) \exp\left(\frac{\beta}{2} - \frac{\beta}{2^{j+1}}\right) \\
&= \frac{e^{\beta/2}}{Z(\beta)} \exp\left(-\frac{\beta}{2^{j+1}} + \log\left(\varepsilon^{3^j} - \varepsilon^{3^{j+2}}\right)\right) \quad (25)
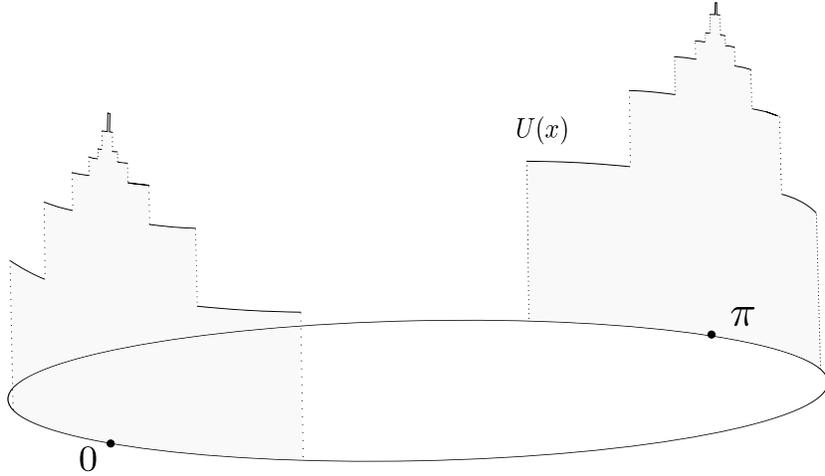\end{aligned}
$$

Figure 1: The graph of the potential U.

Remark 6. Before proceeding to the proof of the proposition we remark that repeated applications of Fubini's Theorem show that the partition function in the volume Λ_n for the potential A satisfies

$$Z_{\Lambda_n}^{\beta, \Phi} = Z(\beta)^{2n}.$$

Proof of Proposition. Let $j \in \mathbb{N}$ and $k \in \{-n, \ldots, n-1\}$. By definition we have

$$\mu_{\Lambda_n}^{\beta,\Phi}(\{\theta_k - \theta_{k+1} \in A_j\}) =$$

$$= \frac{1}{Z_{\Lambda_n}^{\beta,\Phi}} \int_{(0,2\pi]^{2n+1}} \mathbb{1}_{B_{k,j}}(\theta_{-n}, \ldots, \theta_n) \exp\left(\beta \sum_{s=-n}^{n-1} U(\theta_s - \theta_{s+1})\right) \prod_{i=-n}^{n} d\nu(\theta_i).$$

Using the properties of the exponential function, we have that the above integral is given by

$$\int_{(0,2\pi]^{2n+1}} \mathbb{1}_{B_{k,j}}(\theta) \prod_{s=-n}^{n-1} \exp\left(\beta U(\theta_s - \theta_{s+1})\right) \prod_{i=-n}^{n} d\nu(\theta_i). \tag{26}$$

To simplify the exposition we suppose that $k = -n$. The following explanation can easily be modified to work in the general case just by reordering the terms, which can be done by Fubini's Theorem. In the case $k = -n$ it follows from Fubini's Theorem that (26) is equal to

$$\int_{(0,2\pi]^{2n}} \mathbb{1}_{B_{-n,j}}(\theta) \prod_{s=-n}^{n-2} e^{\beta U(\theta_s - \theta_{s+1})} \left(\int_{(0,2\pi]} e^{\beta U(\theta_{n-1} - \theta_n)} d\nu(\theta_n)\right) \prod_{i=-n}^{n-1} d\nu(\theta_i).$$

By the periodicity of U it follows that the integral in parenthesis is independent of θ_{n-1} and equal to $Z(\beta)$. Proceeding by induction, we can see that the above expression simplifies to

$$(Z(\beta))^{2n-1} \int_{(0,2\pi]^2} \mathbb{1}_{B_{-n,j}}(\theta) e^{\beta U(\theta_{-n} - \theta_{-n+1})} d\nu(\theta_{-n}) d\nu(\theta_{-n+1}).$$

To evaluate this, we consider the iterated integral where the most internal integral is made in the variable θ_{-n}, with θ_{-n+1} fixed. For any fixed value of θ_{-n+1}, whenever $\theta \in B_{-n,j}$ we have that $\theta_{-n} \in A_j + \theta_{-n+1}$. In this set the potential U is constant, i.e.,

$$U(\theta_{-n} - \theta_{-n+1}) = c_j.$$

From these observations the previous integral is simply

$$(Z(\beta))^{2n-1} \int_{(0,2\pi]} \int_{A_j + \theta_{-n+1}} e^{\beta c_j} d\nu(\theta_{-n}) d\nu(\theta_{-n+1}),$$

which is equal to

$$(Z(\beta))^{2n-1} e^{\beta c_j} \int_{(0,2\pi]} \int_{A_j + \theta_{-n+1}} d\nu(\theta_{-n}) d\nu(\theta_{-n+1}).$$

Finally by the translation invariance property of the Lebesgue measure we end up with

$$(Z(\beta))^{2n-1} e^{\beta c_j} \nu(A_j). \tag{27}$$

40

Dividing this value by the partition function, we get

$$\mu_{\Lambda_n}^{\beta,\Phi}(\{\theta_k - \theta_{k+1} \in A_j\}) = \frac{\nu(A_j)}{Z(\beta)} e^{\beta c_j}.$$

Note that for $|k| < n$, this expression does not depend on n. From this follows easily that for $|k| < n$ and $j \in \mathbb{N}$,

$$\mu_{\Lambda_n}^{\beta,\Phi}(\{\theta_k - \theta_{k+1} \in A_j\}) = \mu_{\beta A}(\{\theta_k - \theta_{k+1} \in A_j\}).$$

$\square$

Proof of Corollary.

Follows from the fact that, if $j = 2i$ and $x \in A_{2i}$ then

$$
U(x) = \sum_{l=1}^{i} \frac{3}{2^{2l+1}} = \frac{3}{8} \sum_{l=0}^{i-1} \frac{1}{4^l} \;=\; \frac{3}{8}\left(\frac{1 - \frac{1}{4^i}}{1 - \frac{1}{4}}\right)
$$

$$
=\; \frac{1}{2}\left(1 - \frac{1}{4^i}\right) = \frac{1}{2} - \frac{1}{2^{2i+1}} = \frac{1}{2} - \frac{1}{2^{j+1}}.
$$

For the other hand, if $j = 2i + 1$ and $x \in A_{2i+1}$ we have that

$$
U(x) = \frac{1}{4} + \sum_{l=1}^{i} \frac{3}{2^{2l+2}} \;=\; \frac{1}{4} + \frac{3}{16} \sum_{l=0}^{i-1} \frac{1}{2^{2l}}
$$

$$
=\; \frac{1}{4} + \frac{3}{16}\left(\frac{1 - \frac{1}{4^i}}{1 - \frac{1}{4}}\right)
$$

$$
=\; \frac{1}{4} + \frac{1}{4}\left(1 - \frac{1}{4^i}\right) = \frac{1}{2} - \frac{1}{2^{2i+2}} = \frac{1}{2} - \frac{1}{2^{j+1}}.
$$

$\square$

6.3   Maximizing $\mu_{\beta A}(B_{k,j})$

Now we will present some useful calculations to compute $\mu_{\beta A}(B_{k,j})$. We point out that we will need in the future just the case $k = 0$.

Let

$$f_\beta(x) := -\frac{\beta}{2^{x+1}} + \log\left(\varepsilon^{3^x} - \varepsilon^{3^{x+2}}\right).$$

41

The maximum of this function can be found by derivation with respect to x.

$$
\begin{aligned}
f'_\beta(x) &= -\frac{d}{dx}\frac{\beta}{2^{x+1}} + \frac{d}{dx}\log\left(\varepsilon^{3^x} - \varepsilon^{3^{x+2}}\right) \\[2mm]
&= \frac{\beta\log 2}{2^{x+1}} + \frac{\varepsilon^{3^x}(\log\varepsilon\log 3)3^x - 9\varepsilon^{3^{x+2}}(\log\varepsilon\log 3)3^x}{\varepsilon^{3^x} - \varepsilon^{3^{x+2}}} \\[2mm]
&= \frac{\beta\log 2}{2^{x+1}} + (\log\varepsilon\log 3)3^x\left(\frac{\varepsilon^{3^x} - 9\varepsilon^{3^{x+2}}}{\varepsilon^{3^x} - \varepsilon^{3^{x+2}}}\right) \\[2mm]
&= \frac{\beta\log 2}{2^{x+1}} + (\log\varepsilon\log 3)3^x\left(\frac{\varepsilon^{3^x} - 9\varepsilon^{3^{x+2}}}{\varepsilon^{3^x} - \varepsilon^{3^{x+2}}}\right).
\end{aligned}
$$

If x is large enough the equation $f'(x) = 0$ is solvable and the solution is implicitly given by

$$
0 = \frac{\beta\log 2}{2^{x+1}} + (\log\varepsilon\log 3)3^x\left(\frac{\varepsilon^{3^x} - 9\varepsilon^{3^{x+2}}}{\varepsilon^{3^x} - \varepsilon^{3^{x+2}}}\right),
$$

which is equivalent to

$$
\beta = 6^x\frac{-2\log\varepsilon\log 3}{\log 2}\left(\frac{\varepsilon^{3^x} - 9\varepsilon^{3^{x+2}}}{\varepsilon^{3^x} - \varepsilon^{3^{x+2}}}\right). \tag{28}
$$

The fraction appearing in the above equation can be rewritten as

$$
\theta(\epsilon, x) \equiv \frac{\varepsilon^{3^x}}{\varepsilon^{3^x}}\frac{1 - 9\varepsilon^{(3^{x+2}-3^x)}}{1 - \varepsilon^{(3^{x+2}-3^x)}} = \frac{1 - 9\varepsilon^{8\cdot 3^x}}{1 - \varepsilon^{8\cdot 3^x}}. \tag{29}
$$

6.4   An important Lemma

In this subsection we present an important lemma that will help us to estimate the probability $\mu_{\beta A}(B_{k,j})$.

Lemma 31. *Let $(\Omega, \mathcal{B})$ be a measurable space and $(C_j)_{j\in\mathbb{N}}$ a measurable partition of Ω. For any positive β let $\mathbb{P}_\beta$ be a probability measure in $(\Omega, \mathcal{B})$ such that*

$$
\mathbb{P}_\beta(C_j) = \frac{1}{\bar{Z}(\beta)}\exp\left(-\frac{\beta}{2^{j+1}} + \log\left(\varepsilon^{3^j} - \varepsilon^{3^{j+2}}\right)\right)
$$

where $\bar{Z}(\beta)$ is a normalizing constant and $\varepsilon > 0$. Given $\delta > 0$, there exist an $\varepsilon_\delta > 0$ such that, for any $0 < \epsilon < \epsilon_\delta$, for all $j \in \mathbb{N}$, we have

$$
\mathbb{P}_{\beta_j}(C_j) > 1 - \delta, \quad \text{where } \beta_j \text{ is given by}
$$

$$
\beta_j = 6^j 2C_\varepsilon\frac{\log 3}{\log 2}\theta(\varepsilon, j) \quad \text{with} \quad \theta(\varepsilon, j) = \frac{1 - 9\varepsilon^{8\cdot 3^j}}{1 - \varepsilon^{8\cdot 3^j}} \quad \text{and} \quad C_\varepsilon = -\log\varepsilon.
$$

Remark. A more careful analysis of the proof presented here shows that the above lemma also works for a slightly different potential U, where we replace in the initial definition the terms 2^{j+1} and 3^j, by $(1+\delta)^{j+1}$ and $(1+\gamma)^j$, respectively, given that $0 < \delta < \gamma$.

Proof.

Note that $\theta(\varepsilon, x)$ is an increasing function of x, and has limit equal to 1 when $\varepsilon \to 0$ or $x \to +\infty$. Consider the function

$$
\begin{aligned}
f_\beta(x) &= -\frac{\beta}{2^{x+1}} + \log\left(\varepsilon^{3^x} - \varepsilon^{3^{x+2}}\right) \\
&= -\frac{\beta}{2^{x+1}} - C_\varepsilon 3^x + \log\left(1 - \varepsilon^{8.3^x}\right).
\end{aligned}
\tag{30}
$$

From (28) and (29) it follows that its critical point x_0 has to satisfy

$$
\beta = 6^{x_0} 2 C_\varepsilon \frac{\log 3}{\log 2} \theta(\varepsilon, x_0)
\tag{31}
$$

Note that the last equation allows us to obtain the maximum point x_0 of f_β, thus making $x_0 = x_0(\beta)$ an increasing (therefore invertible) and unbounded function of β. Arguing in the inverse direction, for each $x_0 = j_0 \in \mathbb{N}$ we can choose $\beta = \beta(j_0)$ as the unique solution to (31), which means j_0 is the maximum point of $f_{\beta(j_0)}$.

Fix now $j_0 \in \mathbb{N}$. If we set

$$
\kappa_\varepsilon^x = \log\left(1 - \varepsilon^{8.3^x}\right),
$$

(note that κ_ε^x is an increasing function of x) it follows from (30) and (31) that, for any $k \in \mathbb{Z}$

$$
\begin{aligned}
f_{\beta(j_0)}(j_0 + k) &= -\frac{\beta(j_0)}{2^{j_0+k+1}} - C_\varepsilon 3^{j_0+k} + \kappa_\varepsilon^{j_0+k} \tag{32} \\
&= -6^{j_0} 2 C_\varepsilon \frac{\log 3}{\log 2} \cdot \frac{\theta(\varepsilon, j_0)}{2^{j_0+k+1}} - C_\varepsilon 3^{j_0+k} + \kappa_\varepsilon^{j_0+k} \\
&= -3^{j_0}\left[C_\varepsilon \frac{\log 3}{\log 2} \cdot \frac{\theta(\varepsilon, j_0)}{2^k} + C_\varepsilon 3^k\right] + \kappa_\varepsilon^{j_0+k}. \tag{33}
\end{aligned}
$$

Now we will use these identities to get an upper bound for $\frac{\mathbb{P}_{\beta(j_0)}(C_{j_0+k})}{\mathbb{P}_{\beta(j_0)}(C_{j_0})}$. Before going to the upper bound computations we prove:

Identity 1. For any integer $k \geq -j_0 + 1$, we get from (33) the following identity

$$\frac{\mathbb{P}_{\beta(j_0)}(C_{j_0+k})}{\mathbb{P}_{\beta(j_0)}(C_{j_0})} = \frac{e^{f_{\beta(j_0)}(j_0+k)}}{e^{f_{\beta(j_0)}(j_0)}} =$$

$$\exp\left(-3^{j_0}\left[C_\varepsilon \frac{\log 3}{\log 2} \cdot \frac{\theta(\varepsilon,j_0)}{2^k} + C_\varepsilon 3^k\right] + 3^{j_0}\left[C_\varepsilon \frac{\log 3}{\log 2} \cdot \theta(\varepsilon,j_0) + C_\varepsilon\right] + \kappa_\varepsilon^{j_0+k} - \kappa_\varepsilon^{j_0}\right) =$$

$$\exp\left(-3^{j_0}C_\varepsilon\left[\frac{\log 3}{\log 2} \cdot \frac{\theta(\varepsilon,j_0)}{2^k} + 3^k - \frac{\log 3}{\log 2} \cdot \theta(\varepsilon,j_0) - 1\right] + \kappa_\varepsilon^{j_0+k} - \kappa_\varepsilon^{j_0}\right) =$$

$$\exp\left(-3^{j_0}C_\varepsilon\left[\frac{\log 3}{\log 2} \cdot \left(\frac{\theta(\varepsilon,j_0)}{2^k} - \theta(\varepsilon,j_0)\right) - 1 + 3^k\right] + \kappa_\varepsilon^{j_0+k} - \kappa_\varepsilon^{j_0}\right) =$$

$$\exp\left(-3^{j_0}C_\varepsilon 3^k\right)\exp\left(-3^{j_0}C_\varepsilon\left[\frac{\log 3}{\log 2} \cdot \left(\frac{\theta(\varepsilon,j_0)}{2^k} - \theta(\varepsilon,j_0)\right) - 1\right] + \kappa_\varepsilon^{j_0+k} - \kappa_\varepsilon^{j_0}\right) =$$

$$\exp\left(-3^{j_0}C_\varepsilon 3^k\right)\exp\left(3^{j_0}C_\varepsilon\left[\frac{\log 3}{\log 2}\theta(\varepsilon,j_0) \cdot \left(1 - \frac{1}{2^k}\right) + 1\right] + \kappa_\varepsilon^{j_0+k} - \kappa_\varepsilon^{j_0}\right).$$

With the above identities we are ready to show how to get the upper bounds for $\frac{\mathbb{P}_{\beta(j_0)}(C_{j_0+k})}{\mathbb{P}_{\beta(j_0)}(C_{j_0})}$. This will be done by considering separate cases, whether k is positive or negative.

Case $k > 0$. In this case, using the previous identity, $\theta(\varepsilon, j_0) < 1$ and $\kappa_\varepsilon^{j_0+k} - \kappa_\varepsilon^{j_0} < 1$, we have

$$\frac{\mathbb{P}_{\beta(j_0)}(C_{j_0+k})}{\mathbb{P}_{\beta(j_0)}(C_{j_0})} < \exp\left(-3^{j_0}C_\varepsilon 3^k\right)\exp\left(3^{j_0}C_\varepsilon\left[\frac{\log 3}{\log 2} + 1\right] + 1\right)$$

$$\leq \exp\left(-3^{j_0}C_\varepsilon\left[3^k - \frac{\log 3}{\log 2} - 1\right] + 1\right).$$

Of course the above inequality implies, for all $k \in \mathbb{N}$, that

$$\mathbb{P}_{\beta(j_0)}(C_{j_0+k}) \leq \mathbb{P}_{\beta(j_0)}(C_{j_0})\exp\left(1 - 3^{j_0}C_\varepsilon\left[3^k - \frac{\log 3}{\log 2} - 1\right]\right)$$

and then summing over k we obtain

$$\sum_{k=1}^{\infty}\mathbb{P}_{\beta(j_0)}(C_{j_0+k}) \leq \sum_{k=1}^{\infty}\exp\left(1 - 3^{j_0}C_\varepsilon\left[3^k - \frac{\log 3}{\log 2} - 1\right]\right).$$

In order to bound this series, we decompose it as follows

$$\exp\left(1 - 3^{j_0}C_\varepsilon\left[2 - \frac{\log 3}{\log 2}\right]\right) + \sum_{k=2}^{\infty}\exp\left(1 - 3^{j_0}C_\varepsilon\left[3^k - \frac{\log 3}{\log 2} - 1\right]\right).$$

By a simple induction process one proves that $k \leq 3^k - \frac{\log 3}{\log 2} - 1$, for all $k \geq 2$. From

44

this observation it follows the upper bound

$$
\begin{aligned}
\sum_{k=1}^{\infty} \mathbb{P}_{\beta(j_0)}(C_{j_0+k}) &\leq \exp\left(1 - 3^{j_0}C_{\varepsilon}\left[2 - \frac{\log 3}{\log 2}\right]\right) + \sum_{k=2}^{\infty} \exp\left(1 - 3^{j_0}C_{\varepsilon}k\right) \\
&= \exp\left(1 - 3^{j_0}C_{\varepsilon}\left[2 - \frac{\log 3}{\log 2}\right]\right) + \frac{\exp\left(1 - 3^{j_0}2C_{\varepsilon}\right)}{1 - \exp\left(-3^{j_0}C_{\varepsilon}\right)} \\
&\leq \exp\left(1 - 3C_{\varepsilon}\left[2 - \frac{\log 3}{\log 2}\right]\right) + \frac{\exp\left(1 - 6C_{\varepsilon}\right)}{1 - \exp\left(-3C_{\varepsilon}\right)}.
\end{aligned}
$$

As $C_{\varepsilon} = -\log \varepsilon \to \infty$ when $\varepsilon \to 0$, we can choose an ε_0 such that for all $0 < \varepsilon < \varepsilon_0$, we have

$$
\exp\left(1 - 3C_{\varepsilon}\left[2 - \frac{\log 3}{\log 2}\right]\right) + \frac{\exp\left(1 - 6C_{\varepsilon}\right)}{1 - \exp\left(-3C_{\varepsilon}\right)} < \frac{\delta}{2}. \tag{34}
$$

Note that $\varepsilon_0 > 0$ does not depend on j_0.

This implies that

$$
\sum_{k=1}^{\infty} \mathbb{P}_{\beta(j_0)}(C_{j_0+k}) < \frac{\delta}{2}, \tag{35}
$$

for any $j_0 \in \mathbb{N}$, provided $0 < \varepsilon < \varepsilon_0$.

Case $k < 0$. From Identity 1, we have $\frac{\mathbb{P}_{\beta(j_0)}(C_{j_0+k})}{\mathbb{P}_{\beta(j_0)}(C_{j_0})}$ is equal to

$$
\exp\left(-3^{j_0}C_{\varepsilon}3^k\right)\exp\left(3^{j_0}C_{\varepsilon}\left[\frac{\log 3}{\log 2}\theta(\varepsilon, j_0)\cdot\left(1 - \frac{1}{2^k}\right) + 1\right] + \kappa_{\varepsilon}^{j_0+k} - \kappa_{\varepsilon}^{j_0}\right).
$$

Note that we can choose $0 < \epsilon_1 \leq \epsilon_0$ such that, for all $0 < \varepsilon < \epsilon_1$ and all $j_0 \geq 1$, we have

$$
\theta(\varepsilon, j_0)\frac{\log 3}{\log 2} - 1 = \frac{1 - 9\varepsilon^{8\cdot3^{j_0}}}{1 - \varepsilon^{8\cdot3^{j_0}}}\frac{\log 3}{\log 2} - 1 > \frac{\frac{\log 3}{\log 2} - 1}{2} \equiv A. \tag{36}
$$

As a consequence we have

$$
\theta(\varepsilon, j_0) > \frac{\log 2}{\log 3}. \tag{37}
$$

Then

$$
\frac{\log 3}{\log 2}\theta(\varepsilon, j_0)\cdot\left(1 - \frac{1}{2^k}\right) + 1 < 0
$$

for any $k \in \{-j_0 + 1, \ldots, -1\}$, and we have the following inequality, when we use $\kappa_{\varepsilon}^{j_0+k} - \kappa_{\varepsilon}^{j_0} < 0$ and $-3^{j_0}C_{\varepsilon}3^k < 0$

$$
\exp\left(-3^{j_0}C_{\varepsilon}3^k\right)\exp\left(3^{j_0}C_{\varepsilon}\left[\frac{\log 3}{\log 2}\theta(\varepsilon, j_0)\cdot\left(1 - \frac{1}{2^k}\right) + 1\right] + \kappa_{\varepsilon}^{j_0+k} - \kappa_{\varepsilon}^{j_0}\right) \leq
$$

45

$$\leq \exp\left(3C_\varepsilon \left[\frac{\log 3}{\log 2}\theta(\varepsilon, j_0) \cdot \left(1 - \frac{1}{2^k}\right) + 1\right]\right).$$

From this we obtain

$$\mathbb{P}_{\beta(j_0)}(C_{j_0+k}) \leq \exp\left(3C_\varepsilon \left[\frac{\log 3}{\log 2}\theta(\varepsilon, j_0) \cdot \left(1 - \frac{1}{2^k}\right) + 1\right]\right).$$

Using this upper bound, (36) and (37) again, it follows that

$$
\begin{aligned}
\sum_{k=1}^{j_0-1} \mathbb{P}_{\beta(j_0)}(C_{j_0-k}) &\leq \sum_{k=1}^{\infty} \exp\left(3C_\varepsilon \left[\frac{\log 3}{\log 2}\theta(\varepsilon, j_0) \cdot \left(1 - 2^k\right) + 1\right]\right) \\
&< \exp\left(3C_\varepsilon \left[-\frac{\log 3}{\log 2}\theta(\varepsilon, j_0) + 1\right]\right) + \sum_{k=2}^{\infty} \exp\left(3C_\varepsilon \left[\left(1 - 2^k\right) + 1\right]\right) \\
&< e^{-3C_\varepsilon A} + \sum_{k=2}^{\infty} \exp\left(3C_\varepsilon \left(2 - 2^k\right)\right) \\
&< e^{-3C_\varepsilon A} + \sum_{k=2}^{\infty} \exp\left(-3C_\varepsilon k\right) = e^{-3C_\varepsilon A} + \frac{e^{-6C_\varepsilon}}{1 - e^{-3C_\varepsilon}}
\end{aligned}
$$

Using again that $C_\varepsilon = -\log\varepsilon \to +\infty$ when $\varepsilon \to 0$, and $A = \frac{\log 3 - \log 2}{2\log 2} > 0$ we can choose $0 < \varepsilon_\delta \leq \varepsilon_1$ such that for all $0 < \varepsilon < \varepsilon_1$ we have

$$e^{-3\frac{\log 3 - \log 2}{2\log 2}C_\varepsilon} + \frac{e^{-6C_\varepsilon}}{1 - e^{-3C_\varepsilon}} < \frac{\delta}{2}, \tag{38}$$

which implies

$$\sum_{k=1}^{j_0-1} \mathbb{P}_{\beta(j_0)}(C_{j_0-k}) < \frac{\delta}{2}. \tag{39}$$

Finally by (35) and (39) we get

$$\sum_{k \in \mathbb{N}\backslash\{j_0\}} \mathbb{P}_{\beta(j_0)}(C_k) < \delta.$$

if $\epsilon < \epsilon_\delta$. $\qquad\square$

6.5   The non-selection theorem

Now we are ready to state and prove the main result of this section which is due to A. C. D. van Enter and W. M. Ruszel [20]. Note that in the notation we used before the maximizing value is $m(A) = \sup U$.

Theorem 32. *For the potential A described above, consider the family of probability measures $\mu_{\beta A}$, with $\beta \in \mathbb{R}$. Then, in the weak* topology, there is no selection of measure, that is, there is no limit for $\mu_{\beta A}$, when $\beta \to \infty$.*

Proof. Consider the Borel set

$$B = \{(\theta_0 - \theta_1) \in [0, \pi] \subset \mathbb{S}^1\} \subset \mathcal{B}_i,$$

and, the non-continuous function I_B. Given small δ and ϵ, we can approximate I_B by a continuous function $\varphi : \mathcal{B}_i \to \mathbb{R}$, where the set of points where $\varphi \neq I_B$ is contained in the small set

$$\mathcal{D} = \{(\theta_0 - \theta_1) \in [0, \epsilon] \cup [\pi - \epsilon, \pi] \subset \mathbb{S}^1\} \subset \mathcal{B}_i.$$

From the above we can choose a suitable φ, and, also present two sequences s_n and t_n, converging to infinity, such that

$$\int \varphi \, d\mu_{s_n A} > 1 - \delta$$

and

$$\int \varphi \, d\mu_{t_n A} < \delta.$$

This shows that there is no limit for $\mu_{\beta A}$.

$\square$

Remark 7. We point out that the example described above can be adapted in order to produce a continuous potential A which does not select in the limit when $\beta \to \infty$ [20].

References

[1] T. Bousch, La condition de Walters. *Ann. Sci. ENS*, 34, (2001)

[2] V. Bangert, Mather sets for twist maps and geodesics on tori, Dynamics Reported 1, 1-56, 1988.

[3] A. Baraviera, A. O. Lopes and Ph. Thieullen, A Large Deviation Principle for Gibbs states of Holder potentials: the zero temperature case. *Stoch. and Dyn.* (6), 77-96, (2006).

[4] A. Baraviera, R. Leplaideur and A. O. Lopes, Selection of measures for a potential with two maxima at the zero temperature limit, to appear in SIAM Journ. of Appl Dyn.

[5] A. Baraviera, A. O. Lopes A and J. Mengue, On the selection of subaction and measure for a subclass of Walters's potentials, preprint UFRGS (2011)

[6] P. Bernard and G. Contreras. *A Generic Property of Families of Lagrangian Systems.* Annals of Math. Vol. 167, No. 3, 2008

[7] R. Bowen, Gibbs States and the Ergodic Theory of Anosov Diffeomorphisms, Lecture notes in Math., volume 470, Springer-Verlag, 1975.

[8] J. Brémont, Gibbs measures at temperature zero. *Nonlinearity*, 16(2): 419–426, 2003.

[9] J.R. Chazottes and M. Hochman, On the zero-temperature limit of Gibbs states, *Commun. Math. Phys.*, Volume 297, N. 1, 2010

[10] J.R. Chazottes, J.M. Gambaudo and E. Ugalde, Zero-temperature limit of one dimensional Gibbs states via renormalization: the case of locally constant potentials. *Erg. Theo. and Dyn. Sys.* (2010).

[11] J. Conway, A Course in Functional Analysis, Springer Verlag, 1990

[12] G. Contreras and R. Iturriaga. *Global minimizers of autonomous Lagrangians*, 22° Colóquio Brasileiro de Matemática, IMPA, 1999.

[13] W. Chou and R. J. Duffin, An additive eigenvalue problem of physics related to linear programming, *Advances in Applied Mathematics* **8** (1987), 486-498.

[14] W. Chou and R. Griffiths, *Ground states of one-dimensional systems using effective potentials*, Physical Review B, Vol. 34, N 9, 6219-6234, 1986

[15] G. Contreras, A. O. Lopes and Ph. Thieullen. Lyapunov minimizing measures for expanding maps of the circle, Ergodic Theory and Dynamical Systems Vol 21, 1379-1409, 2001.

[16] G. Contreras, A. O. Lopes and E. R. Oliveira, Ergodic Transport Theory, periodic maximizing probabilities and the twist condition, preprint UFRGS (2011)

[17] K. Deimling. *Nonlinear Functional Analysis*, Springer Verlag, 1985

[18] A. Dembo and O. Zeitouni, Large Deviations Techniques and Applications, Springer Verlag, 1998.

[19] A. C. D. van Enter, R. Fernandez and A. D. Sokal, Regularity properties and pathologies of position-space renormalization-group transformations: Scope and limitations of Gibbsian theory, *Journ. of Stat. Phys.* V.72, N 5/6 879-1187, (1993)

[20] A. C. D. van Enter and W. M. Ruszel, Chaotic Temperature Dependence at Zero Temperature, Journal of Statistical Physics, Vol. 127, No. 3, 567-573, (2007)

[21] A. Fathi, Théorème KAM faible et théorie de Mather sur les systèmes lagrangiens, Comptes Rendus de l'Académie des Sciences, Série I, Mathématique Vol 324 1043-1046, 1997.

[22] Y. Fukui and M. Horiguchi, One-dimensional Chiral *XY* Model at finite temperature, Interdisciplinary Information Sciences, Vol 1, 133-149, N. 2 (1995)

[23] E. Garibaldi and A. O. Lopes, On Aubry-Mather theory for symbolic Dynamics, Ergodic Theory and Dynamical Systems, Vol 28 , Issue 3, 791-815 (2008)

[24] E. Garibaldi and A. O. Lopes, The effective potential and transshipment in thermodynamic formalism at temperature zero, preprint UFRGS (2010)

[25] E. Garibaldi and Ph. Thieullen, Minimizing orbits in the discrete Aubry-Mather model. Nonlinearity 24 (2011), no.2, 563-611.

[26] E. Garibaldi and Ph. Thieullen, Description of some ground states by Puiseux technics, preprint, 2010

[27] H.-O. Georgii, Gibbs Measures and Phase Transitions. de Gruyter, Berlin, (1988).

[28] D. A. Gomes, Viscosity solution method and the discrete Aubry-Mather problem, *Discrete and Continuous Dynamical Systems, Series A* **13** (2005), 103-116.

[29] D. A. Gomes and E. Valdinoci, Entropy Penalization Methods for Hamilton-Jacobi Equations, Adv. Math. 215, No. 1, 94-152, 2007.

[30] D. A. Gomes, A. O. Lopes and J. Mohr, The Mather measure and a large deviation principle for the entropy penalized method. Commun. Contemp. Math. 13 (2011), no.2, 235-268.

[31] R.B. Israel, Convexity in the theory of lattice gases, Princeton University Press, 1979.

[32] O. Jenkinson. Ergodic optimization, Discrete and Continuous Dynamical Systems, Series A, V. 15, 197-224, 2006

[33] G. Keller, Gibbs States in Ergodic Theory, Cambridge Press, 1998.

[34] S. Karlin. *Total Positivity*. Stanford Univ. Press, 1968.

[35] J. Lebowitz and A. Martin-Löf, On the Uniqueness of the Gibbs State for Ising Spin Systems, Comm. Math. Phys. **25**, 276-282 (1972).

[36] R. Leplaideur, A dynamical proof for the convergence of Gibbs measures at temperature zero, *Nonlinearity*, 18(6):2847–2880, 2005.

[37] A. O. Lopes, Entropy and Large Deviation, NonLinearity, Vol. 3, N. 2, 527-546, 1990.

[38] A. O. Lopes, J. Mohr, R. Souza and Ph. Thieullen, Negative entropy, zero temperature and stationary Markov chains on the interval, *Bulletin of the Brazilian Mathematical Society* **40** (2009), 1-52.

[39] A. O. Lopes and J. Mengue, Zeta measures and Thermodynamic Formalism for temperature zero, *Bulletin of the Brazilian Mathematical Society* 41 (3) pp 449-480 (2010)

[40] A. O. Lopes and J. Mengue, Selection of measure and a Large Deviation Principle for the general one-dimensional XY model, preprint UFRGS (2011)

[41] A. O. Lopes, E. R. Oliveira and Ph. Thieullen, The dual potential, the involution kernel and transport in ergodic optimization, *preprint*, 2008.

[42] R. Mañé. *Generic properties and problems of minimizing measures of Lagrangian systems*, Nonlinearity, Vol 9, 273-310, 1996.

[43] J. Mather, Action minimizing invariant measures for positive definite Lagrangian Systems, Math. Z., 207 (2), pp 169-207, 1991

[44] D. H. Mayer, *The Ruelle-Araki transfer operator in classical statistical mechanics*, LNP 123, Springer Verlag 1980

[45] W. Parry and M. Pollicott. *Zeta functions and the periodic orbit structure of hyperbolic dynamics*, Astérisque Vol 187-188 1990

[46] M. Pollicott and M. Yuri, Dynamical systems and Ergodic Theory, Cambrige Press, 1998

[47] R. T. Rockafellar, Extension of Fenchel's duality theorem for convex functions, Duke Math. J., 33, 81-89, 1966.

[48] D. Ruelle, Thermodynamic Formalism, second edition, Cambridge, 2004.

[49] H. H. Schaefer. *Banach Lattices and Positive Operators*, Springer Verlag, 1974.

[50] B. Simon, The Statistical Mechanics of Lattice Gases, Princeton Univ Press, 1993

[51] R. R. Souza, Sub-actions for weakly hyperbolic one-dimensional systems, *Dynamical Systems* 18 **(2)**, 165-179 (2003).

[52] Y. Velenik, Phase Separation as a Large Deviations Problem. Phd Thesis. Lausanne. (2003).

[53] O. Sarig, Lecture Notes on Thermodynamic Formalism for Topological Markov Shifts, preprint PenState USA (2009)

[54] A. Taylor, Introduction to Functional Analysis, Krieger Pub Co, (1986).

[55] F. Spitzer. A Variational characterization of finite Markov chains. The Annals of Mathematical Statistics. (43): N.1 303-307, 1972.

[56] A. N. Shiryaev, Probability. Second Edition, Springer (1984).